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The Phase Space Structure of Multi Particle Models in 2+1 Gravity

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Abstract

What can we learn about quantum gravity from a simple toy model, without actually quantizing it? The toy model consists of a finite number of point particles, coupled to three dimensional Einstein gravity. It has finitely many physical degrees of freedom. These are basically the relative positions of the particles in spacetime and the conjugate momenta. The resulting reduced phase space is derived from Einstein gravity as a topological field theory. The crucial point is thereby that we do not make any a priori assumptions about this phase space, except that the dynamics of the gravitational field is defined by the Einstein Hilbert action. This already leads to some interesting features of the reduced phase space, such as a non-commutative structure of spacetime when the model is quantized.

Outline and summary

It is widely believed that in a quantum theory of gravity, the concept of a smooth spacetime manifold has to be replaced by something new, perhaps a kind of *non-commutative* spacetime, whatever this precisely means. In a non-commutative spacetime the position of a particle in space at a moment of time is not a well defined point on a manifold. There is some kind of uncertainty relation, which makes it impossible for the particle to be localized. The spacetime itself obtains a foamy structure at small length and time scales. It is no longer possible to measure, for example, the relative position of two objects with arbitrary precision. The length scale at which this is expected to happen is the *Planck scale*, which is about $\ell_{\mathrm{Pl}} \approx 10^{-33}$cm.

The purpose of the present article is to study a toy model, which is based on Einstein gravity in three spacetime dimensions. There are no local excitations of the gravitational field, and in particular no gravitational waves is a three dimensional spacetime [1, 2]. But nevertheless it is possible to define non-trivial dynamical systems. The special feature of these toy models is that they have finitely many physical degrees of freedom, while at the level of Einstein gravity as a field theory they share all those feature with the full theory in higher dimensions which are interesting from the point of view of quantum gravity. These features are, in particular, the diffeomorphism group of the spacetime manifold, which appears as a gauge group, and the presence of a dimensionful coupling constant, which is Newton's constant G.

There are two classes of such toy models. The *topological* models are those that do not include matter. Physical degrees of freedom only arise if the universe has a non-trivial topology [3, 4]. The *particle* models contain pointlike particles as matter sources. The physical degrees of freedom are then basically the relative positions and momenta of the particles in spacetime [5, 6]. There are many interesting and unexpected features of such particle models, some of them even related to time machines and black hole physics [7, 8, 9, 10]. In the context of quantum gravity, the basic idea behind the models is to use a pointlike particle to *probe* the structure of the spacetime at small length scales [11, 12, 13]. If quantum gravity sets some principle limitation on the possible length scales that can be resolved, then this should be seen when the toy model is quantized.

In three spacetime dimensions, Newton's constant G is an inverse mass or energy, in units where the velocity of light is equal to one. Its inverse defines the *Planck mass* M_{Pl}, which is a classical constant in the sense that $\hbar$ is not involved in the definition. So, there is a natural energy scale in the classical theory of general relativity. Another special feature of three dimensional gravity is that it admits pointlike matter sources, which are not hidden in black holes. The gravitational field of a point particle with mass m is simply a cone with a deficit angle of $8\pi Gm$ [1, 5]. The spacetime in the neighbourhood of the world line is the direct product of this cone with a real line.

Due to this conical structure of the spacetime, there is a gravitational interaction between particles, even though there are no local gravitational forces acting on the particles. Each world line is just a timelike or lightlike geodesics, thus a straight line in an otherwise flat spacetime. However due to the non-trivial global behaviour of a geodesic on a cone, the particles are effectively attracted by each other when they pass each other. But actually, we are here not interested in these *dynamical* features the particles. There are various methods to solve the classical equations of motion, and to obtain an overview of all possible spacetimes [5, 6, 14].

What we are rather interested in are the *kinematical* features of the particle models. By this we mean, for example, the phase space structures, and in particular the way gravity influences the symplectic structure and the Poisson brackets. It is this what distinguishes gravity from other typical interactions that can be defined between point particles. It turns out that gravity not only enters the Hamiltonian, by adding some kind of interaction potential or whatsoever, but also the symplectic structure. When the model is quantized, this has an immediate influence on the commutation relations between certain operators, long before the actual dynamics is imposed by the Hamiltonian. So, we expect the principle effects of quantum gravity at show up at this kinematical level.

It is therefore not of much help to know the classical solutions to the equations of motion, hence in this case Einstein's equations in a three dimensional spacetime. Instead, we have to define a proper action functional for the particle model, and derive from this the phase space structures. As we are dealing with a toy model that has no counterpart in the real world, we have to be very careful with this definition, if we want to learn something about quantum gravity and the way it influences the small scale structure of spacetime. In particular, we should not make any a priori assumptions, which might be motivated by the expected results. The *only* assumption that we are going to make is that the kinematical and dynamical features of the gravitational field are completely specified by the Einstein Hilbert action.

The particle model is in fact well defined as a dynamical system at the level of Einstein gravity as a field theory on a fixed spacetime manifold. We shall show this in section 4. At this level, the definition of the particle model has more or less become standard [14, 15, 16]. However, there are then various different ways to proceed. The model is defined as a field theory, where the basic variables are the metric or the dreibein and the spin connection on a given spacetime. So it has an infinite dimensional configuration space, though only finitely many physical degrees of freedom. One can use the ADM formulation of general relativity [17], to set up a Hamiltonian framework. But then one still has an infinite dimensional phase space.

To this one can apply the usual phase space reduction methods. There are constraints and associated gauge symmetries, and these can be solved and divided out. What remains is a finite dimensional reduced phase space, which is basically spanned by the position and momentum coordinates of the particles. A possible way to derive this reduced phase space, which is motivated by the conventional methods for gauge field theories, is to impose a gauge condition directly on the basic fields, hence the metric. In the ADM formulation of general relativity, this means that the foliation of the spacetime is chosen in a particular way, introducing a globally defined absolute time and space. The gauge fixed metric on the space at a moment of time is then specified by finitely many variables, and roughly speaking they define the positions and velocities of the particles in space at that moment of time [6, 14, 15, 16].

So far, this approach is completely straightforward. However, it turns out to be quite hard to return to physics after this mathematical framework has been set up. The reason is the following. By construction, the reduced phase space is covered by a global canonical coordinate chart. One has a Poisson bracket which is diagonal in the basic variables, and the Hamiltonian is some function of these variables. Formally, a state can be specified by fixing n points on a plane and their velocities, where n is the number of particles. However, this plane and the points on the plane have not very much to do with the real space in which the particles are moving. The real space at a moment of time is a conical surface with n tips, which is embedded into a locally flat

spacetime with n conical singularities.

The actual geometry of this conical space is encoded in the locations and velocities of the points on the plane in a somewhat implicit way. Consider for example the following question. Given two particles, what is the distance of these particles in spacetime, at a fixed moment of time? The spacetime is locally flat, and thus a geodesic connecting two particles is just a straight line. There might be several such geodesics in a conical spacetime, but then we can choose one and ask for its length. Expressing this length as a phase space function is an interesting question, because it is the corresponding quantum operator which tells us something about the spacetime structure when the model is quantized. But unfortunately, in the gauge fixed formulation this simple quantity is a very complicated function of the basic phase space variables.

To know the geodesic distance between two particles, we first have to know the spacetime metric in terms of the coordinates used, and to know this we have to take into account all particles and their relative motion. Hence, a very simple object, the flat spacetime metric, is encoded in a complicated way. The idea of this article is therefore to present an alternative phase space reduction, which leads to a different set of basic phase space variables. The price that we have to pay is that these coordinates do not longer provide a global chart on the reduced phase space. Instead, it will be covered by an atlas of finitely many local coordinate charts. But on the other hand, these coordinates will have an immediate geometric and thus physical interpretation.

To explain this, it is convenient first to consider the limit $G \to 0$, where the gravitational interaction is switched off. In this case, the particle model reduces to a special-relativistic free particle model. The particles are moving freely in a flat three dimensional Minkowski space. The phase space of this model is spanned by the *absolute* positions and momenta of the particles with respect to the embedding Minkowski space, which serves as a reference frame. But it is also possible to introduce *relative* coordinates and momenta, and to define a phase space spanned by them. This will be shown in section 1. The idea is, in a sense, to use the free particle model as a much simpler and well known toy model for our real toy model, and to show the analogy to the ADM, or Hamiltonian framework of general relativity.

We can think of the flat Minkowski space in which the particles are living as a spacetime manifold, which is foliated by a family of spacelike slices. They are labeled by a kind of ADM time coordinate, and the state at a moment of time is specified by the geometry of the respective slice. A typical such slice is shown in figure 1. Its geometry is, on the other hand, specified by a set of relative position and momentum coordinates for the particles. They refer to the links of a *triangulation*. The time evolution of this geometry is provided by the mass shell constraints for the particles, in analogy of the Hamiltonian constraints of general relativity.

So far, this is just a strange way to describe a very simply system on n free relativistic point particles. However, the remarkable feature of this description is that we only have to make a few modifications, or *deformations*, in order to describe the geometry of the spacetime of the interacting system. This will be the subject of section 2, and the resulting phase space structures will be discussed in section 3. In a sense, we can *switch on* the gravitational interaction in a continuous way, using Newton's constant G as a deformation parameter. First we have to cut the polygons of the triangulation apart, as indicated in figure 2, then they are deformed, as shown in figure 3, and finally we can glue them together again. The result is a spacelike surface which is no longer globally embedded into Minkowski space.

Instead, it is a conical surface with n tips. This is the space at a moment of time with the grav-

itational interaction between the particles switched on. The crucial point is thereby that the link variables, hence those phase space variables that define the geometry of the individual polygons, are still the same as before. In particular, they still represent the same physical quantities, for example the relative position of two particles in spacetime. This is very different to the gauge fixed method, where the phase space variables are also the usual position and momentum coordinates of the free particles in the limit $G \to 0$. But when gravity is switched on, the physical interpretation of the variables is lost.

The only problem is that it is now no longer possible to use these geometric variables as global coordinates on the phase space. There are some principle obstacles, and therefore it is not possible to have at the same time a global coordinate chart on the phase space, and a straightforward geometric interpretation of the coordinates [18]. So, we have to cover the phase space by an atlas of local coordinate charts. But as long as we are interested in the local features of the phase space, and the symplectic structure is a local object, we can stick to one of the charts and need not care about the global structure of the phase space.

It is at this point, where the perhaps most interesting feature arise, namely the *non-commutative* structure of spacetime. Let us briefly explain what happens. For a pair of free particles in flat Minkowski space, we can introduce a vector z^a ($a = 0, 1, 2$), representing the relative position of the particles in spacetime. Clearly, the Poisson brackets for the components of this vector are zero. Such a vector can still be defined when gravity is switched on, and it still has the same physical interpretation as a relative position of two particles in spacetime. However, now it turns out that the Poisson brackets and therefore the resulting quantum commutators are given by

$$\{z^a, z^b\} = 8\pi G\, \varepsilon^{abc}\, z_c \quad \Rightarrow \quad [z^a, z^b] = 8\pi\mathrm{i}\, \ell_{\mathrm{Pl}}\, \varepsilon^{abc}\, z_c,$$

where $\ell_{\mathrm{Pl}} = G\hbar$ is again the *Planck length*, and ε^{abc} is the Levi Civita tensor. It defines the structure constants for the three dimensional Lorentz group. So, we have in this case a well defined notion of what is meant by a non-commutative spacetime. The position coordinates of the particle no longer commute. And obviously this non-commutative structure disappears in the limit $G \to 0$.

The same quantum commutator has also been found for a much simpler single particle model [13]. The only difference was that there the vector z^a did not represent the relative position of two particles, but the absolute position of the particle with respect to a reference frame. Unfortunately, there was some ambiguity in the definition of the reference frame, and we had to make some additional assumptions, beyond the basic one about the Einstein Hilbert action. One could therefore argue a little bit about the result, and the suggestion was that these problems could be overcome when going over from a single to a multi particle model.

And in fact, we shall here see that the non-commutative structure arises from the phase space reduction applied to the Einstein Hilbert action without any further assumption. The reason why the multi particle model is, in a sense, better defined than a single particle model has to do with the asymptotic structure of a three dimensional spacetime. To set up a proper Hamiltonian framework of general relativity, one has to impose a fall off condition on the metric at infinity. This is typically some kind of asymptotical flatness condition. Moreover, one has to introduce a *reference frame* at spatial infinity, which can be interpreted as the rest frame of some external observer.

The phase space and the Hamiltonian are well defined only if such a reference frame is in-

cluded into the configuration space of the model [19]. The Killing symmetries of the asymptotic metric are then of special interest. They represent the possible translations and rotations of the universe with respect to the reference frame. In the Hamiltonian framework, these are the rigid symmetries of the model, and the associated conserved charges represent the total momentum and angular momentum of the universe. Now, the special feature of Einstein gravity in three dimensions is the conical structure of spacetime, which also manifests itself at spatial infinity. The universe is not asymptotically flat, but asymptotically conical.

The symmetry group of a cone, however, is smaller than the symmetry group of a flat Minkowski space. The only symmetries are time translations and spatial rotations, whereas boosts and spatial translations are not allowed. This restricted symmetry group is that of the *centre of mass frame* of a system of point particles. It is therefore appropriate to choose the reference frame so that it coincides with the centre of mass frame. But on the other hand, this does not make sense for a single particle model, because there is then no physical degree of freedom left. And this was the reason for the various difficulties and ambiguities to arise, which were related to the definition of the reference frame.

All these ambiguities disappear when we define the multi particle model so that the reference frame coincides with the centre of mass frame of the universe. In section 4 we shall show this explicitly, defining the model in the first order formalism of general relativity, using the dreibein and the spin connection. We'll see that the required boundary terms to be added to the Einstein Hilbert action, defining the reference frame at spatial infinity, are uniquely fixed without making any further assumptions, in contrast to the single particle model where this was not the case. But in all other aspect the multi particle model will be a straightforward generalization of the single particle model.

In particular, the coupling of the particles to the gravitational field will be defined in the same way, by formally converting the matter degrees of freedom into topological degrees of freedom, and introducing generalized, group valued momenta of the particles. As all this is explained in detail in [13]. So, we shall here only briefly sketch this. The actual subject of section 4 is then to perform the phase space reduction, and to derive all those phase space structures which we have to define without further motivation in section 3. The logical order of the article is actually so that section 4 comes first, then section 2 describes the model from the spacetime point of view, section 3 takes the phase space point of view, and finally in section 1 we take the limit $G \to 0$.

There are then various ways to proceed, but this is not any more the subject of this article. It actually serves a rather technical purpose, namely to set up a proper classical Hamiltonian framework for the particle model, which is as general as possible, but also as simple as possible, in the sense that no further assumption enters the model beyond the basic one mentioned above about the Einstein Hilbert action. The results can therefore serve as a starting point for a further analysis after some simplifications are made. For example, the simplest case is the *Kepler system* with $n = 2$ particles. This can be solve completely both at the classical level, deriving the trajectories in the phase space, and at the quantum level, solving the Schrödinger, or actually a generalized Klein Gordon equation [20, 21].

And for this particular example, we find indeed some interesting features of quantum gravity. It is no longer possible to localize the particles in space, or to bring the particles closer to each other than a certain distance, which is of the order of the Planck length. All this can be seen very explicitly and clearly when the model is quantized canonically. It is even possible to derive

the wave functions explicitly and see how the energy eigenstates of look like, when expressed in terms of physical variables. Hence the states can directly be interpreted as probability amplitudes, and we see explicitly how quantum gravity restrict the ability to probe small length scales.

Another way to proceed further might be to consider the low coupling limit, in which case it could be possible to solve and quantize the n particle system exactly, or to consider a closed universe with a few particles as a new kind of cosmological toy model. But whatever one does in this direction, *solving* always means more then just writing down the solutions to the equations of motion or the quantum states in terms of *some* variables. This tells us almost nothing about the physics. To learn something about quantum gravity one has to give the solutions, hence the classical trajectories or the quantum states, in terms of *physical* variables. And for this purpose the Hamiltonian framework presented here might be useful.

1 Free particles

Consider a very simple system of $n \geq 2$ free relativistic point particles, living in a three dimensional, flat Minkowski space. As a vector space, we identify this with the spinor representation $\mathfrak{sl}(2)$ of the three dimensional Lorentz algebra. The position variables $\boldsymbol{x}_\pi = x_\pi^a \boldsymbol{\gamma}_a$ and the momentum vectors $\boldsymbol{p}_\pi = p_\pi^a \boldsymbol{\gamma}_a$ of the particles are represented as traceless 2×2 matrices, with $\boldsymbol{\gamma}_a$ ($a = 0, 1, 2$) being the orthonormal basis of the usual gamma matrices (A.1). A collection of definitions and formulas regarding the vector and matrix notation can be found in the appendix. The index π is used to label the individual particles.

We have a $6n$ dimensional phase space which is spanned by the variables $\boldsymbol{x}_\pi$ and $\boldsymbol{p}_\pi$. The symplectic potential and the resulting Poisson brackets are

$$\Theta = \sum_\pi \tfrac{1}{2}\mathrm{Tr}(\boldsymbol{p}_\pi \, \mathrm{d}\boldsymbol{x}_\pi) \quad \Rightarrow \quad \{p_\pi^a, x_\pi^b\} = \eta^{ab}. \tag{1.1}$$

Note that we sometimes switch between the matrix and vector notation, and use whatever is more convenient. The Hamiltonian is a linear combination of the mass shell constraints $\mathcal{C}_\pi$, with Lagrange multipliers $\zeta_\pi \in \mathbb{R}$ as coefficients,

$$\mathcal{H} = \sum_\pi \zeta_\pi \, \mathcal{C}_\pi, \qquad \mathcal{C}_\pi = \tfrac{1}{4}\mathrm{Tr}(\boldsymbol{p}_\pi^2) + \tfrac{1}{2}m_\pi^2 \approx 0. \tag{1.2}$$

The physical phase space is the subspace defined by the mass shell constraints and the positive energy conditions,

$$\tfrac{1}{2}\mathrm{Tr}(\boldsymbol{p}_\pi^2) = -m_\pi^2, \qquad p_\pi^0 = \tfrac{1}{2}\mathrm{Tr}(\boldsymbol{p}_\pi \boldsymbol{\gamma}^0) > 0. \tag{1.3}$$

The mass shell constraints are first class constraints, and the associated gauge symmetries are the reparameterizations of the world lines as functions of a common, unphysical time parameter t. The gauge freedom corresponds to the freedom to choose the multipliers in the time evolution equations,

$$\dot{\boldsymbol{p}}_\pi = \{\mathcal{H}, \boldsymbol{p}_\pi\} = 0, \qquad \dot{\boldsymbol{x}}_\pi = \{\mathcal{H}, \boldsymbol{x}_\pi\} = \zeta_\pi \, \boldsymbol{p}_\pi. \tag{1.4}$$

The dot denotes the derivative with respect to t. The free particle system is a simple toy model for the ADM formulation of general relativity. The coordinate t on the world lines is the globally defined, but unphysical ADM time. The mass shell constraints $\mathcal{C}_\pi$ are the Hamiltonian

constraints. The multipliers ζ_π represent the lapse function. And the gauge symmetries, the reparameterizations of the world lines, are the spacetime diffeomorphisms. To make this analogy even closer, we impose the following two restrictions on the phase space variables. The first is a gauge restriction. It is analogous to the restriction to spacelike foliations in general relativity. The second restriction is a kind of asymptotical flatness condition, although at the moment this is not immediately obvious.

The gauge restriction is as follows. At each moment of ADM time t, the particles must be located on a common spacelike surface in the embedding Minkowski space. In other words, the parameterization of the world lines is induced by a foliation of Minkowski space by a family of spacelike slices. For simplicity, let us further assume that the states where the particles are at the same point in space are excluded from the phase space. As long as we do not study the dynamics, that is the classical trajectories, this is not going to be a problem. The gauge condition is then requires all relative position vectors to be spacelike. Hence,

$$\boldsymbol{x}_{\pi_2} - \boldsymbol{x}_{\pi_1} \quad \text{is spacelike for all } \pi_1, \pi_2. \tag{1.5}$$

With this restriction, the phase space is no longer a vector space. But it is still a $6n$ dimensional manifold, because the set of all spacelike vectors is an open subset of Minkowski space. The number of independent gauge symmetries in not affected by this restriction.

The second restriction has to do with the rigid symmetries. The positions $\boldsymbol{x}_\pi$ and momenta $\boldsymbol{p}_\pi$ are defined with respect to a *reference frame*. We can think of it as the rest frame of some external observer. The rigid symmetries are the translations and Lorentz rotations of the particles with respect to this reference frame. The associated Noether charges are the components of the total momentum and angular momentum vector,

$$\boldsymbol{P} = \sum_\pi \boldsymbol{p}_\pi, \qquad \boldsymbol{J} = \tfrac{1}{2} \sum_\pi [\boldsymbol{p}_\pi, \boldsymbol{x}_\pi]. \tag{1.6}$$

The reference frame coincides with the *centre of mass* frame if

$$\boldsymbol{P} = M\,\boldsymbol{\gamma}_0, \qquad \boldsymbol{J} = S\,\boldsymbol{\gamma}_0, \qquad M, S \in \mathbb{R}, \quad M > 0. \tag{1.7}$$

This is the second restriction that we impose on the phase space variables. In the centre of mass frame, there exists an absolute time and an absolute space. The symmetry group is restricted to the two dimensional subgroup of time translations and spatial rotations. A plane orthogonal to the $\boldsymbol{\gamma}_0$-axis represents an instant of absolute time. The $\boldsymbol{\gamma}_0$-axis is the world line of a *fictitious* centre of mass particle. The total energy M is the mass, and the total angular momentum S is the spin of this particle. These are the conserved charges associated with the remaining symmetries.

Except for one very special situation, a centre of mass frame always exists. If all particles are massless, and if they move with the velocity of light into the same spatial direction, then the total momentum is lightlike. These states are excluded. The dimension of the restricted phase space $6n - 4$, since only the four spatial components of the equations (1.7) are restrictions on the phase space variables. The time components are the definitions of the charges M and S. The gauge symmetries are not affected. We still have the freedom to reparameterize the world lines independently, within the range that is allowed by (1.5).

From the phase space point of view, the restriction to the centre of mass frame provides two pairs of second class constraints, whereas the mass shell constraints are still first class. For a

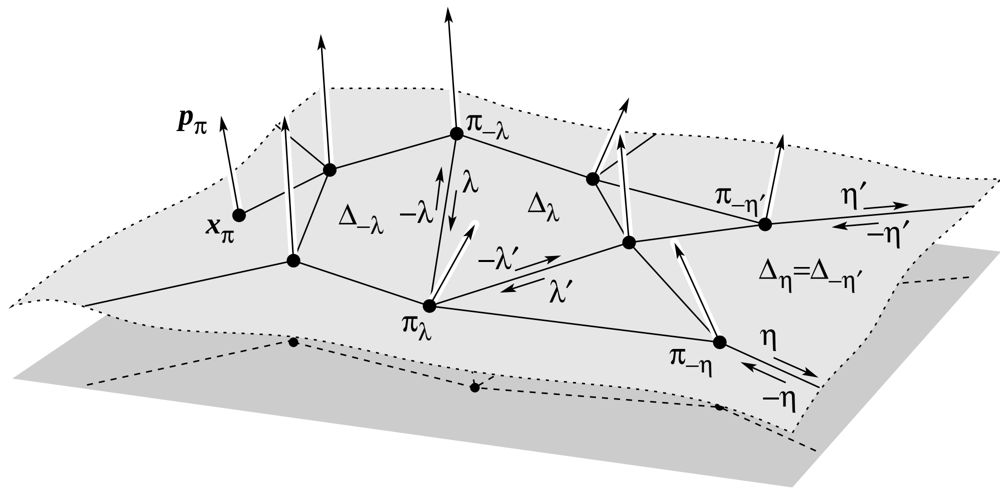

Figure 1: The triangulated ADM surface of the free particle system. The particles are located at the positions $\boldsymbol{x}_\pi$ in the embedding Minkowski space, which defines the centre of mass frame and also the reference frame. The momentum vectors $\boldsymbol{p}_\pi$ are the tangent vectors of the world lines attached to these points. The surface is cut along the links λ of a graph Γ, and divided into a collection Π of simply connected polygons Δ.

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proper Hamiltonian description of the restricted system, we either have to replace the Poisson brackets with Dirac brackets, or we have to eliminate the second class constraints, by going over to a new set of independent phase space variables. This is what we are going to do now. We are looking for a convenient way to parameterize the solutions to the equations $\boldsymbol{P} = M\boldsymbol{\gamma}_0$ and $\boldsymbol{J} = S\boldsymbol{\gamma}_0$, which can later be generalized to describe the interacting particles.

Triangulations

Consider a foliation of the embedding Minkowski space by a family of spacelike slices, labeled by an ADM time coordinate t, and so that the parameterization of the world lines is induced by this foliation. A typical slice is shown in figure 1. Let us call it the *ADM surface* at a moment of time t. Note that we have to keep the ADM time t apart from the absolute time defined by the $\boldsymbol{\gamma}_0$-coordinate in the centre of mass frame. The ADM surface need not be an spatial plane in Minkowski space. For the time being, it can be any spacelike surface. The positions $\boldsymbol{x}_\pi$ of the particles at the given moment of time are the intersections of the world lines with the ADM surface, and the momenta $\boldsymbol{p}_\pi$ are the tangent vectors of the world lines attached to these points.

A *triangulation* is defined by a collection Γ of *oriented links*. Every link $\lambda \in \Gamma$ represents a geodesic in Minkowski space. It either connects two particles or extends from a particle to infinity. The particles are thus the *vertices* of a *graph*, and there is an additional, special vertex at infinity. The graph must be connected, hence there must be at least one link attached to every vertex, and it must be possible to go from every particle to every other along the links. Furthermore, the graph must be spacelike, which means that the ADM surface can be smoothly

deformed, so that all the links are contained in the surface. And finally, let us also require that all the links extending to infinity are *spatial half lines*. A spatial half line is a geodesic which is orthogonal to the γ_0-axis, and extends from some point to infinity.

If this is the case, then the graph Γ divides the ADM surface into a collection Π of simply connected *polygons*. To define the various relations between the vertices, the links, and the polygons, we introduce the following notation. With each link $\lambda \in \Gamma$, we also have the *reversed* link $-\lambda \in \Gamma$. It represents the same geodesic with opposite orientation. We distinguish between *internal* and *external* links. An internal link λ *begins* at a particle $\pi_{-\lambda}$, and it *ends* at a particle π_λ. For the reversed link $-\lambda$, these two particles are interchanged, which explains the notation. The subset of all internal links is called $\Gamma_0 \subset \Gamma$.

It is sometimes useful to assign a preferred orientation to each internal link. For this purpose, we split the set $\Gamma_0 = \Gamma_+ \cup \Gamma_-$ into two disjoint subsets, so that $\lambda \in \Gamma_+$ implies $-\lambda \in \Gamma_-$ and vice versa. Since a priori there is no such preferred orientation, this decomposition is arbitrary. Whenever we use it we have to make sure that the results are independent of the particular decomposition. The external links already have a preferred orientation. We define $\Gamma_\infty \subset \Gamma$ to be the set of all external links oriented towards infinity, and $\Gamma_{-\infty}$ is the set of all reversed external links oriented towards the particles. Again, it is so that $\eta \in \Gamma_\infty$ implies $-\eta \in \Gamma_{-\infty}$ and vice versa.

An external link $\eta \in \Gamma_\infty$ begins at some particle $\pi_{-\eta}$, but it does not end at any particle, and vice versa for the reversed external link $-\eta \in \Gamma_{-\infty}$. We use the symbol λ to denote a link in general or an internal link, whereas the symbol η always denotes an external link. A useful identity is the following decomposition of the graph Γ into four disjoint subsets,

$$\Gamma = \Gamma_+ \cup \Gamma_- \cup \Gamma_\infty \cup \Gamma_{-\infty}, \qquad \Gamma_0 = \Gamma_+ \cup \Gamma_-. \tag{1.8}$$

If π is some vertex, then $\Gamma_\pi = \{\lambda \,|\, \pi_\lambda = \pi\}$ is the set of all links *ending* at π, and $\Gamma_{-\pi} = \{\lambda \,|\, \pi_{-\lambda} = \pi\}$ is the set of all links *beginning* at π. For the special vertex at infinity, this agrees with the definition of Γ_∞ and $\Gamma_{-\infty}$ above. Since every link begins and ends at some vertex, including the one at infinity, we have the following alternative disjoint decomposition of the graph,

$$\Gamma = \Gamma_\infty \cup \bigcup_\pi \Gamma_\pi = \Gamma_{-\infty} \cup \bigcup_\pi \Gamma_{-\pi}. \tag{1.9}$$

It is useful to assign a cyclic ordering to each of these sets. Let π be a particle, and consider a loop surrounding this particle in clockwise direction. The ordering in which the links cross this loop defines the cyclic ordering of the set Γ_π. If $\lambda \in \Gamma_\pi$ is a link ending at π, then the next following link in clockwise direction is called the *successor* of λ, which is denoted by $\lambda' \in \Gamma_\pi$.

If there is only one link attached of a vertex, then we have $\lambda' = \lambda$, otherwise this obviously defines a cyclic ordering. The same applies to the set Γ_∞ of external links, but for consistency this is ordered in counter clockwise direction. Thus every external link $\eta \in \Gamma_\infty$ has a unique successor $\eta' \in \Gamma_\infty$, which is the next external link that we cross when we walk around a circle at infinity in counter clockwise direction. Since every link is contained in exactly one of the subsets Γ_π or Γ_∞, every link also has a unique successor, which is an element of the same subset.

Each pair of links λ and $-\lambda$ represents the common boundary of two polygons Δ_λ and $\Delta_{-\lambda}$. We define Δ_λ to be the polygon that lies *to the left* of λ, and consequently $\Delta_{-\lambda}$ lies *to the right* of λ. For each polygon $\Delta \in \Pi$, the set $\Gamma_\Delta = \{\lambda \,|\, \Delta_\lambda = \Delta\}$ represents the boundary of

the polygon, traversed in counter clockwise direction. A link $\lambda \in \Gamma_\Delta$ is called an *edge* of the polygon Δ. Since every oriented link is an edge of exactly one polygon, we have yet another disjoint decomposition of the graph, namely

$$\Gamma = \bigcup_{\Delta \in \Pi} \Gamma_\Delta. \tag{1.10}$$

There is relation between the cyclically ordered sets Γ_π and Γ_∞, and the sets Γ_Δ of edges. If $\lambda, \lambda' \in \Gamma_\pi$ are two successive links ending at a vertex π, which might also be the one at infinity, then $\lambda, -\lambda' \in \Gamma_\Delta$ are two successive edges of some polygon Δ. For $\lambda, \lambda' \in \Gamma_\pi$ we always have $\Delta_\lambda = \Delta_{-\lambda'}$.

We distinguish between *compact* polygons $\Delta \in \Pi_0$, and *non-compact* polygons $\Delta \in \Pi_\infty$, so that $\Pi = \Pi_0 \cup \Pi_\infty$. A compact polygon $\Delta \in \Pi_0$ is bounded by at least three internal edges $\lambda \in \Gamma_\Delta \subset \Gamma_0$. A non-compact polygon has exactly one external edge $\eta \in \Gamma_\Delta \cap \Gamma_\infty$ oriented towards infinity, exactly one reversed external edge $-\eta' \in \Gamma_\Delta \cap \Gamma_{-\infty}$, and one or more internal edges $\lambda \in \Gamma_\Delta \cap \Gamma_0$. And the links $\eta, \eta' \in \Gamma_\infty$ are thereby always two successive external links. Consequently, there are as many non-compact polygons as there are external links.

There is also a relation between the number of compact polygons and the number of internal links. It follows from Euler's polyhedron formula, and it involves the number of particles. Let $\ell_0 = |\Gamma_+| = |\Gamma_-|$ be the number of internal links, so that each link is only counted with one orientation, and let $\ell_\infty = |\Gamma_\infty| = |\Gamma_{-\infty}|$ be the number of external links. Similarly, $\wp_0 = |\Pi_0|$ is the number of compact polygons, and $\wp_\infty = |\Pi_\infty|$ is the number of non-compact polygons. Then we have the following relations,

$$\ell_0 - \wp_0 = n - 1, \qquad \ell_\infty = \wp_\infty. \tag{1.11}$$

They basically define the topology of the surface which is triangulated. We may release these conditions later on when we consider the coupled system, considering more general spatial topologies. But for the free particles the ADM surface always has the topology of $\mathbb{R}^2$.

Relative momenta

Let us now come back to the phase space of the particles. The primary problem was to solve the second class constraints (1.7). Let us first consider the equation $\boldsymbol{P} = M\boldsymbol{\gamma}_0$, which only involves the momenta of the particles. To find the general solution, we introduce a *relative momentum* vector $\boldsymbol{q}_\lambda \in \mathfrak{sl}(2)$ assigned to every link $\lambda \in \Gamma$, so that

$$\boldsymbol{p}_\pi = \sum_{\lambda \in \Gamma_\pi} \boldsymbol{q}_\lambda, \qquad \boldsymbol{q}_{-\lambda} = -\boldsymbol{q}_\lambda. \tag{1.12}$$

The momentum $\boldsymbol{p}_\pi$ of a particle π is the sum of the relative momenta $\boldsymbol{q}_\lambda$ of all links ending at π. It is useful to think of the vectors $\boldsymbol{q}_\lambda$ as currents flowing through the links. The second equation then states that the current flowing through the reversed link is the reversed current. And the first equation states that the momentum of a particle π is the total current that flows out of the graph at the vertex π. Using this picture, and the fact that the graph Γ is connected, it is easy to show that, for a given set of momenta $\boldsymbol{p}_\pi$, it is always possible to find a suitable set of relative momenta $\boldsymbol{q}_\lambda$, so that (1.12) holds.

However, there is no one-to-one relation between the momenta $\boldsymbol{p}_\pi$ and the relative momenta $\boldsymbol{q}_\lambda$. We are free to add an extra current, which only flows along the boundary of one of the polygons, but never enters or leaves the graph at any vertex. This has no influence on the particles. More explicitly, let $\boldsymbol{\chi}_\Delta \in \mathfrak{sl}(2)$ be a Minkowski vector for each polygon $\Delta \in \mathit{\Pi}$, and consider the following transformation of the relative momenta,

$$\boldsymbol{q}_\lambda \mapsto \boldsymbol{q}_\lambda + \boldsymbol{\chi}_{\Delta_{-\lambda}} - \boldsymbol{\chi}_{\Delta_\lambda}. \tag{1.13}$$

Inserting this into (1.12), we find

$$\boldsymbol{p}_\pi \mapsto \boldsymbol{p}_\pi + \sum_{\lambda\in\mathit{\Gamma}_\pi} (\boldsymbol{\chi}_{\Delta_{-\lambda}} - \boldsymbol{\chi}_{\Delta_\lambda}) = \boldsymbol{p}_\pi + \sum_{\lambda\in\mathit{\Gamma}_\pi} \boldsymbol{\chi}_{\Delta_{-\lambda'}} - \sum_{\lambda\in\mathit{\Gamma}_\pi} \boldsymbol{\chi}_{\Delta_\lambda} = \boldsymbol{p}_\pi. \tag{1.14}$$

For the first equality, we split the sum into two, and then we replaced the index $\lambda \in \mathit{\Gamma}_\pi$ in the first sum by its successor $\lambda' \in \mathit{\Gamma}_\pi$. Thus we shifted the summation index cyclically. But now, remember that for two successive links $\lambda, \lambda' \in \mathit{\Gamma}_\pi$, we always have $\Delta_\lambda = \Delta_{-\lambda'}$. Therefore the two sums are equal and cancel. We find that the momentum vectors are invariant under any transformation of the form (1.13).

Before considering this in more detail, let us show how the link variables can be used to solve the constraint equation $\boldsymbol{P} = M\boldsymbol{\gamma}_0$. The total momentum vector can be written as

$$\boldsymbol{P} = \sum_\pi \boldsymbol{p}_\pi = \sum_\pi \sum_{\lambda\in\mathit{\Gamma}_\pi} \boldsymbol{q}_\lambda = \sum_{\lambda\in\mathit{\Gamma}_+} \boldsymbol{q}_\lambda + \sum_{\lambda\in\mathit{\Gamma}_-} \boldsymbol{q}_\lambda + \sum_{\eta\in\mathit{\Gamma}_{-\infty}} \boldsymbol{q}_\eta = -\sum_{\eta\in\mathit{\Gamma}_\infty} \boldsymbol{q}_\eta. \tag{1.15}$$

The first equality is the definition of $\boldsymbol{P}$, and the second follows from (1.12). To derive the third equality, we used (1.8) and (1.9), which implies

$$\bigcup_\pi \mathit{\Gamma}_\pi = \mathit{\Gamma}_+ \cup \mathit{\Gamma}_- \cup \mathit{\Gamma}_{-\infty}. \tag{1.16}$$

Finally, the last equality in (1.15) follows from the fact that $\boldsymbol{q}_{-\lambda} = -\boldsymbol{q}_\lambda$ for all links $\lambda \in \mathit{\Gamma}$. The sums over $\mathit{\Gamma}_+$ and $\mathit{\Gamma}_-$ cancel, and the sum over $\mathit{\Gamma}_{-\infty}$ can be written as a sum over $\mathit{\Gamma}_\infty$. Now, consider the equation $\boldsymbol{P} = M\boldsymbol{\gamma}_0$. This is obviously satisfied if we choose the relative momentum vectors for the external links to be proportional to $\boldsymbol{\gamma}_0$, hence

$$\boldsymbol{q}_\eta = -M_\eta\,\boldsymbol{\gamma}_0, \qquad \boldsymbol{q}_{-\eta} = M_\eta\,\boldsymbol{\gamma}_0, \qquad \eta \in \mathit{\Gamma}_\infty. \tag{1.17}$$

The variable M_η is called the *energy* of the external link $\eta \in \mathit{\Gamma}_\infty$. The total energy is then the sum of all energies assigned to the external links,

$$\boldsymbol{P} = \sum_{\eta\in\mathit{\Gamma}_\infty} M_\eta\,\boldsymbol{\gamma}_0 \quad \Rightarrow \quad M = \sum_{\eta\in\mathit{\Gamma}_\infty} M_\eta. \tag{1.18}$$

So, what we found is a partly redundant but complete parameterization of the momentum vectors $\boldsymbol{p}_\pi$ of the particles, which are subject to the equation $\boldsymbol{P} = M\boldsymbol{\gamma}_0$. The independent parameters are the relative momentum vectors $\boldsymbol{q}_\lambda$ for $\lambda \in \mathit{\Gamma}_+$, and the energies M_η for $\eta \in \mathit{\Gamma}_\infty$.

The redundancy transformations are still given by (1.13). But there is now a certain restriction on the parameters $\boldsymbol{\chi}_\Delta$. The external relative momenta $\boldsymbol{q}_\eta$ for $\eta \in \mathit{\Gamma}_\infty$ must be proportional to

γ_0, before and after the transformation. This is the case if and only if $\chi_\Delta = -\omega_\Delta \gamma_0$ for all non-compact polygons $\Delta \in \Pi_\infty$, where $\omega_\Delta \in \mathbb{R}$. This implies that the energies transform as

$$M_\eta \mapsto M_\eta + \omega_{\Delta_{-\eta}} - \omega_{\Delta_\eta}. \tag{1.19}$$

As a cross check, let us count the number of independent redundancies. There are three components of the vector χ_Δ for each compact polygon $\Delta \in \Pi_0$, and one real number ω_Δ for each non-compact polygon $\Delta \in \Pi_\infty$. Thus all together we have $3\wp_0 + \wp_\infty$ parameters. However, there is actually one degree of freedom less. If we set $\chi_\Delta = -\omega \gamma_0$ for all polygons $\Delta \in \Pi$, and $\omega_\Delta = \omega$ for all non-compact polygons $\Delta \in \Pi_\infty$, where $\omega \in \mathbb{R}$ is some fixed number, then the transformations (1.13) and (1.19) are void. Therefore, the number of independent redundancies is

$$3\wp_0 + \wp_\infty - 1 = (3\ell_0 + \ell_\infty) - (3n - 2). \tag{1.20}$$

The equality follows from (1.11). The result is what we have to expect. On the right hand side, we have the difference between the number of independent link variables, and the original number of momentum variables. There are $3\ell_0$ independent components of the relative momenta $\boldsymbol{q}_\lambda$ for $\lambda \in \Gamma_+$, and ℓ_∞ independent energies M_η for $\eta \in \Gamma_\infty$. Originally, we had $3n$ independent components of the momentum vectors $\boldsymbol{p}_\pi$, and we imposed two constraints, the spatial components of the equations $\boldsymbol{P} = M\gamma_0$.

Relative positions

The positions $\boldsymbol{x}_\pi$ of the particles are subject to the constraint $\boldsymbol{J} = S\gamma_0$. They can now be parameterized in a similar, but in a sense dual way. For each link $\lambda \in \Gamma$, we introduce yet another vectors $\boldsymbol{z}_\lambda \in \mathfrak{sl}(2)$. For an internal link, it represents the *relative position* of the two particles connected by the link,

$$\boldsymbol{z}_\lambda = \boldsymbol{x}_{\pi_\lambda} - \boldsymbol{x}_{\pi_{-\lambda}} \quad \Rightarrow \quad \boldsymbol{z}_{-\lambda} = -\boldsymbol{z}_\lambda, \qquad \lambda \in \Gamma_0. \tag{1.21}$$

For an external link, it is a spacelike unit vector, which specifies the *direction* of the link in the embedding Minkowski space. Remember that we required the external links to be spatial half lines. The vector $\boldsymbol{z}_\eta$ for $\eta \in \Gamma_\infty$ is therefore a spacelike unit vector which is orthogonal to the γ_0-axis, and the same applies to the vector $\boldsymbol{z}_{-\eta}$, which points into the opposite direction. Using the definition (A.6) we can write

$$\boldsymbol{z}_\eta = \boldsymbol{\gamma}(\phi_\eta), \qquad \boldsymbol{z}_{-\eta} = -\boldsymbol{\gamma}(\phi_\eta), \qquad \eta \in \Gamma_\infty, \tag{1.22}$$

where ϕ_η is the angular direction of the external link η. In figure 2 we have cut the ADM surface apart, and embedded the individual polygons into some auxiliary Minkowski space, to visualize the definition of the vectors $\boldsymbol{z}_\lambda$ for internal and external links.

Finally, there is yet another link variable that can be assigned to every external link $\eta \in \Gamma_\infty$. Consider a *clock* which is located at the far end of the link, and which shows that absolute time in the centre of mass frame. The time shown on this clock is then equal to the γ_0-coordinate of the particle $\pi_{-\eta}$ to which the link is attached, thus

$$T_\eta = x^0_{\pi_{-\eta}}, \qquad \eta \in \Gamma_\infty. \tag{1.23}$$

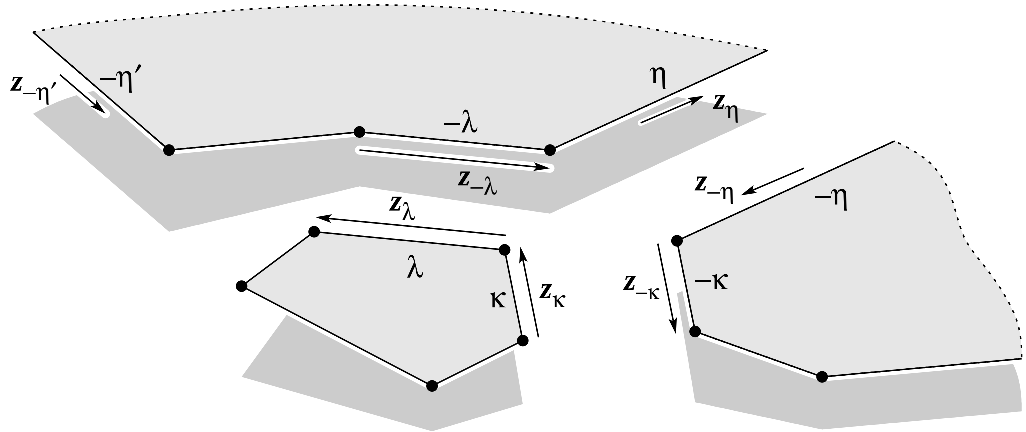

Figure 2: The geometry of the individual polygons is determined, up to smooth deformations, by the relative position vectors $\boldsymbol{z}_\lambda$ for $\lambda \in \Gamma_0$, and the spatial unit vectors $\boldsymbol{z}_\eta$ for $\eta \in \Gamma_{\pm\infty}$. The ADM surface in figure 1 is reconstructed by gluing the polygons together along the edges. The relation $\boldsymbol{z}_{-\lambda} = -\boldsymbol{z}_\lambda$ ensures that they always fit together.

So, we have now assigned a vector $\boldsymbol{z}_\lambda$ to each internal link $\lambda \in \Gamma_0$, and two scalars ϕ_η and T_η to each external link. But these variables are not all independent. It follows from the definition (1.21) that, for every compact polygon, the relative position vectors representing its edges add up to zero,

$$\sum_{\lambda \in \Gamma_\Delta} \boldsymbol{z}_\lambda = 0, \qquad \Delta \in \Pi_0. \tag{1.24}$$

For non-compact polygons, there is a similar relation. Let $\eta, \eta' \in \Gamma_\infty$ be two successive external links, and let $\Delta = \Delta_\eta = \Delta_{-\eta'}$ be the non-compact polygon in between. There is then the following relation between the clocks and the relative position vectors, which follows from (1.21) and (1.23),

$$T_{\eta'} - T_\eta + \sum_{\lambda \in \Gamma_\Delta \cap \Gamma_0} z^0_\lambda = 0, \qquad \Delta \in \Pi_\infty. \tag{1.25}$$

Hence, the difference between the absolute time on the two external edges is equal to the minus the sum of the the time components of the internal edges.

The statement is now the following. Given the vectors $\boldsymbol{z}_\lambda$ for all $\lambda \in \Gamma$, and the clocks T_η for all $\eta \in \Gamma_\infty$, and provided that the *consistency conditions* (1.24) and (1.25) are satisfied, then the positions $\boldsymbol{x}_\pi$ of the particles are uniquely determined by the system of linear equations (1.21) and (1.23), and the constraint equation $\boldsymbol{J} = S\boldsymbol{\gamma}_0$. The proof is not very complicated. Given the vectors $\boldsymbol{z}_\lambda$ for all $\lambda \in \Gamma_0$, can already derive the relative position of any two particles. We just have to walk along the internal links from one particle to the other, and add up the relative position vectors. The relation (1.24) ensures that the result is independent of the path that we took.

So, we already know the absolute positions up to an overall translation $\boldsymbol{x}_\pi \mapsto \boldsymbol{x}_\pi + \boldsymbol{v}$, where $\boldsymbol{v} \in \mathfrak{sl}(2)$ is some unknown Minkowski vector. Now, consider the equation $\boldsymbol{J} = S\boldsymbol{\gamma}_0$. Let us

express the angular momentum vector $\boldsymbol{J}$ in terms of the link variables. Using the definition (1.6), and inserting the representation (1.12) of the momentum vectors, we get

$$\boldsymbol{J} = \tfrac{1}{2} \sum_{\pi} [\boldsymbol{p}_\pi, \boldsymbol{x}_\pi] = \tfrac{1}{2} \sum_{\pi} \sum_{\lambda \in \Gamma_\pi} [\boldsymbol{q}_\lambda, \boldsymbol{x}_{\pi_\lambda}]. \tag{1.26}$$

Here we used that $\pi = \pi_\lambda$ for all $\lambda \in \Gamma_\pi$. It follows from (1.16) that this can be written as

$$\boldsymbol{J} = \tfrac{1}{2} \sum_{\lambda \in \Gamma_+} [\boldsymbol{q}_\lambda, \boldsymbol{x}_{\pi_\lambda}] + \tfrac{1}{2} \sum_{\lambda \in \Gamma_-} [\boldsymbol{q}_\lambda, \boldsymbol{x}_{\pi_\lambda}] + \tfrac{1}{2} \sum_{\eta \in \Gamma_{-\infty}} [\boldsymbol{q}_\eta, \boldsymbol{x}_{\pi_\eta}]. \tag{1.27}$$

The first two sums can be combined, using $\boldsymbol{q}_{-\lambda} = -\boldsymbol{q}_\lambda$, and then we can insert the definition (1.21) of $\boldsymbol{z}_\lambda$. To simplify the last sum, we use the definition (1.17) of M_η. The result is

$$\boldsymbol{J} = \tfrac{1}{2} \sum_{\lambda \in \Gamma_+} [\boldsymbol{q}_\lambda, \boldsymbol{z}_\lambda] + \tfrac{1}{2} \sum_{\lambda \in \Gamma_\infty} M_\eta \, [\boldsymbol{\gamma}_0, \boldsymbol{x}_{\pi_{-\eta}}]. \tag{1.28}$$

Now, consider an overall translation $\boldsymbol{x}_\pi \mapsto \boldsymbol{x}_\pi + \boldsymbol{v}$. The first term is obviously invariant, and so the angular momentum vector transforms as

$$\boldsymbol{J} \mapsto \boldsymbol{J} + \tfrac{1}{2} M \, [\boldsymbol{\gamma}_0, \boldsymbol{v}]. \tag{1.29}$$

This implies that the spatial components of the unknown vector $\boldsymbol{v}$ are determined by the equation $\boldsymbol{J} = S\boldsymbol{\gamma}_0$. But then we still have the freedom to perform a time translation $\boldsymbol{x}_\pi \mapsto \boldsymbol{x}_\pi + v\boldsymbol{\gamma}_0$. To fix the remaining degree of freedom, we need to know the absolute time coordinate of at least one particle. This additional information is provided by the clocks T_η. It is sufficient to know one of them, but it does not matter which one. The consistency condition (1.25) ensures that we always get the same results.

So, all together we find that the link variables provide sufficient information to reconstruct the original position and momentum variables $\boldsymbol{x}_\pi$ and $\boldsymbol{p}_\pi$ of the particles, which are subject to the constraints $\boldsymbol{P} = M\boldsymbol{\gamma}_0$ and $\boldsymbol{J} = S\boldsymbol{\gamma}_0$. The charges M and S can then also be expressed in terms of the link variables. According to (1.18) and (1.28) we have

$$M = \sum_{\eta \in \Gamma_\infty} M_\eta, \qquad S = \sum_{\lambda \in \Gamma_+} L_\lambda, \quad \text{where} \quad L_\lambda = \tfrac{1}{4} \mathrm{Tr}(\, [\boldsymbol{q}_\lambda, \boldsymbol{z}_\lambda] \, \boldsymbol{\gamma}^0). \tag{1.30}$$

The total energy is, in a sense, distributed over the external links, and the total angular momentum is distributed over the internal links. Note that $L_{-\lambda} = L_\lambda$, which implies that the sum is independent of the decomposition $\Gamma_0 = \Gamma_+ \cup \Gamma_-$.

But still, the parameterization is not unique. We still have the redundancy transformation (1.13) and (1.19), acting on the relative momenta and energies. Moreover, we never used the unit vectors $\boldsymbol{z}_{\pm\eta}$ for $\eta \in \Gamma_\infty$ when we derived the absolute positions $\boldsymbol{x}_\pi$ of the particles. We are therefore free to perform the following additional redundancy transformations,

$$\phi_\eta \mapsto \phi_\eta + \epsilon_\eta, \qquad \eta \in \Gamma_\infty, \tag{1.31}$$

where $\epsilon_\eta \in \mathbb{R}$ are free parameters. This is the freedom that we also have when we introduce the triangulation in figure 1. The directions of the external links are not fixed, so within a certain

range we can rotated them. This requires a smooth deformation of the ADM surface, but it does not affect the positions or the momenta of the particles.

Taking this into account, we have the following total number of redundancies,

$$3\wp_0 + \wp_\infty + \ell_\infty - 1. \tag{1.32}$$

On the other hand, we also have the consistency conditions to be satisfied by the link variables. There are $3\wp_0$ components of the vector equations (1.24), and $\wp_\infty$ scalar equations (1.25). But these are not all independent. Let us add the $\boldsymbol{\gamma}_0$-components of all equations (1.24), and all equations (1.25). Then the clocks drop out, and what remains is the sum of the $\boldsymbol{\gamma}_0$-components of all vectors $\boldsymbol{z}_\lambda$ for $\lambda \in \Gamma_0$. But this sum is zero, because for each link $\lambda \in \Gamma_0$, there is also a reversed link $-\lambda \in \Gamma_0$. So, the total number of consistency conditions is $3\wp_0 + \wp_\infty - 1$.

The idea is to consider these consistency conditions as first class constraints, and the redundancy transformations as the associated gauge symmetries. To treat the rotations (1.31) of the external links in the same way, we shall add an auxiliary *angular momentum* L_η for $\eta \in \Gamma_\infty$ to the link variables, which is canonically conjugate to the direction ϕ_η, and impose the constraint $L_\eta = 0$, which then generates the gauge transformation (1.31). With this extension of the phase space variables, the number of consistency conditions, or constraints, becomes equal to the number of redundancies, or gauge symmetries (1.32).

The extended phase space

Let us summarize all this in the following definition of an *extended phase space* $\mathcal{Q}$, for a system of n free point particles in the centre of mass frame. Since there are finitely many possible ways to define a graph Γ, the extended phase space $\mathcal{Q}$ consists of a finite number of disconnected components $\mathcal{Q}_\Gamma$. There is one component for each possible graph Γ with n particles. We shall discuss these global features of the phase space $\mathcal{Q}$ in a moment. Let us first consider a fixed graph Γ, and define the component $\mathcal{Q}_\Gamma$. It is spanned by the following *link variables*,

- a *relative momentum vector* $\boldsymbol{q}_\lambda \in \mathfrak{sl}(2)$ and a *relative position vector* $\boldsymbol{z}_\lambda \in \mathfrak{sl}(2)$ for every internal link $\lambda \in \Gamma_0$, and
- an *energy* M_η, a *clock* T_η, an *angular momentum* L_η, and a *direction* ϕ_η for every external link.

The vectors are not all independent. For the internal links we have

$$\boldsymbol{q}_{-\lambda} = -\boldsymbol{q}_\lambda, \qquad \boldsymbol{z}_{-\lambda} = -\boldsymbol{z}_\lambda, \qquad \lambda \in \Gamma. \tag{1.33}$$

The independent phase space variables are thus $\boldsymbol{q}_\lambda$ and $\boldsymbol{z}_\lambda$ for $\lambda \in \Gamma_+$, where $\Gamma_0 = \Gamma_+ \cup \Gamma_-$ can be any possible decomposition. The relation (1.33) is also valid for external links, if we define the associated vectors to be

$$\boldsymbol{q}_\eta = -\boldsymbol{q}_{-\eta} = -M_\eta\,\boldsymbol{\gamma}_0, \qquad \boldsymbol{z}_\eta = -\boldsymbol{z}_{-\eta} = \boldsymbol{\gamma}(\phi_\eta), \qquad \eta \in \Gamma_\infty. \tag{1.34}$$

The symplectic structure on $\mathcal{Q}$ is defined by the following canonical symplectic potential,

$$\Theta = \sum_{\eta\in\Gamma_\infty} (T_\eta\,\mathrm{d}M_\eta + L_\eta\,\mathrm{d}\phi_\eta) - \tfrac{1}{2}\sum_{\lambda\in\Gamma_+} \mathrm{Tr}(\mathrm{d}\boldsymbol{q}_\lambda\,\boldsymbol{z}_\lambda). \tag{1.35}$$

Of course, there is no particular motivation for this at the moment. And in fact, we must show that this coincides with the original symplectic structure (1.1). Otherwise the definition does not make sense. We still want to consider the same dynamical system. The proof will be given further below, so let us for the moment assume that the definition is correct. We can then read off the following non-vanishing Poisson brackets. For each external link, we have two canonical pairs

$$\{T_\eta, M_\eta\} = 1, \qquad \{L_\eta, \phi_\eta\} = 1, \qquad \eta \in \varGamma_\infty. \tag{1.36}$$

For each internal link $\lambda \in \varGamma_0$, the vectors $\boldsymbol{q}_\lambda$ and $\boldsymbol{z}_\lambda$ are canonically conjugate to each other. However, we have to take into account the relations (1.33), stating that the vectors for λ and $-\lambda$ are not independent. What we find is

$$\{q^a_\lambda, z^b_\lambda\} = \eta^{ab} \quad \Rightarrow \quad \{q^a_{-\lambda}, z^b_\lambda\} = \{q^a_\lambda, z^b_{-\lambda}\} = -\eta^{ab}, \qquad \lambda \in \varGamma_0. \tag{1.37}$$

All other brackets are zero, in particular those between variables assigned to different links. We should also note that the expression (1.35) for the symplectic potential is independent of the decomposition $\varGamma_0 = \varGamma_+ \cup \varGamma_-$. We have to sum over all internal links with some preferred orientation, but it does not matter how this orientation is chosen. If we replace λ with $-\lambda$, both $\boldsymbol{q}_\lambda$ and $\boldsymbol{z}_\lambda$ change their sign, and thus the corresponding contribution to $\varTheta$ is invariant. The same holds for the Poisson brackets (1.37).

Kinematical constraints

How is this extended phase space $\mathcal{Q}$ related to the original phase space of the free particles, restricted to the centre of mass frame? We have seen that the link variables are subject to some consistency conditions. The idea is to impose them as *kinematical constraints* on the extended phase space $\mathcal{Q}$. Thus, for each compact polygon, there is a vector valued constraint (1.24),

$$\boldsymbol{Z}_\Delta = \sum_{\lambda\in\Delta} \boldsymbol{z}_\lambda \approx 0, \qquad \Delta \in \varPi_0. \tag{1.38}$$

For each non-compact polygon, there is a scalar constraint (1.25),

$$\mathcal{Z}_\Delta = T_{\eta'} - T_\eta + \sum_{\lambda\in\varGamma_\Delta\cap\varGamma_0} z^0_\lambda \approx 0, \qquad \Delta \in \varPi_\infty. \tag{1.39}$$

And finally, for each external link we impose a constraint which eliminates the artificially introduced angular momentum L_η,

$$\mathcal{J}_\eta = L_\eta \approx 0, \qquad \eta \in \varGamma_\infty. \tag{1.40}$$

The subspace $\mathcal{S}_\varGamma \subset \mathcal{Q}_\varGamma$ defined by these constraints is called the *kinematical phase space*.

It is then straightforward to check that the kinematical constraints are first class, and that the associated gauge symmetries are the previously considered redundancy transformations. It is convenient first to define a general linear combination. We introduce a set of Lagrange multipliers, a vector $\boldsymbol{\chi}_\Delta \in \mathfrak{sl}(2)$ for each compact polygon $\Delta \in \varPi_0$, a scalar $\omega_\Delta \in \mathbb{R}$ for each non-compact polygon $\Delta \in \varPi_\infty$, and another scalar $\epsilon_\eta \in \mathbb{R}$ for each external link $\eta \in \varGamma_\infty$. Then we define

$$\mathcal{K} = \tfrac{1}{2}\sum_{\Delta\in\varPi_0} \mathrm{Tr}(\boldsymbol{\chi}_\Delta \boldsymbol{Z}_\Delta) + \sum_{\Delta\in\varPi_\infty} \omega_\Delta\, \mathcal{Z}_\Delta + \sum_{\eta\in\varGamma_\infty} \epsilon_\eta\, \mathcal{J}_\eta. \tag{1.41}$$

Using the definition of the constraints, and rearranging the summation, this can also be written as

$$\mathcal{K} = \tfrac{1}{2} \sum_{\lambda \in \Gamma_0} \mathrm{Tr}(\boldsymbol{\chi}_{\Delta_\lambda} \boldsymbol{z}_\lambda) + \sum_{\eta \in \Gamma_\infty} (\omega_{\Delta_{-\eta}} - \omega_{\Delta_\eta})\, T_\eta + \sum_{\eta \in \Gamma_\infty} \epsilon_\eta \, L_\eta. \tag{1.42}$$

Here we set $\boldsymbol{\chi}_\Delta = -\omega_\Delta \boldsymbol{\gamma}_0$ for $\Delta \in \Pi_\infty$, so that the first sum also contains the contributions from the non-compact polygons. It is then very easy to derive the following non-vanishing Poisson brackets,

$$\begin{aligned} \{\mathcal{K}, \boldsymbol{q}_\lambda\} &= \boldsymbol{\chi}_{\Delta_{-\lambda}} - \boldsymbol{\chi}_{\Delta_\lambda}, & & \lambda \in \Gamma_0, \\ \{\mathcal{K}, M_\eta\} &= \omega_{\Delta_{-\eta}} - \omega_{\Delta_\eta}, \qquad \{\mathcal{K}, \phi_\eta\} = \epsilon_\eta, & & \eta \in \Gamma_\infty. \end{aligned} \tag{1.43}$$

These are obviously the infinitesimal generators of the redundancy transformations (1.13), (1.19), and (1.31). We can also see that the constraints are not all independent. Setting $\epsilon_\eta = 0$, $\omega_\Delta = \omega$, and $\boldsymbol{\chi}_\Delta = -\omega\, \boldsymbol{\gamma}_0$ for some fixed $\omega \in \mathbb{R}$, we find that the linear combination (1.42) vanishes identically, again because the sum over all vectors $\boldsymbol{z}_\lambda$ for $\lambda \in \Gamma_0$ is zero. We have the following relation between the constraints,

$$\sum_{\Delta \in \Pi_0} Z^0_\Delta + \sum_{\Delta \in \Pi_\infty} \mathcal{Z}_\Delta = 0. \tag{1.44}$$

Everything fits together consistently. We can also recover the original phase space spanned by the absolute positions $\boldsymbol{x}_\pi$ and momentum vectors $\boldsymbol{p}_\pi$ of the particles. It is the quotient space $\mathcal{S}_\Gamma/\!\sim$, where two states $\Phi_1, \Phi_2 \in \mathcal{S}_\Gamma$ are equivalent, $\Phi_1 \sim \Phi_2$, if they can be transformed into each other by a redundancy transformation generated by the constraints.

We can now also check that the symplectic potential (1.35) coincide with the original one on the kinematical subspace $\mathcal{S}_\Gamma$. We start from the original expression (1.1), and insert the definition (1.12) of the momentum vectors $\boldsymbol{p}_\pi$,

$$\Theta = \tfrac{1}{2} \sum_{\pi} \mathrm{Tr}(\boldsymbol{p}_\pi \, \mathrm{d}\boldsymbol{x}_\pi) = \tfrac{1}{2} \sum_{\pi} \sum_{\lambda \in \Gamma_\pi} \mathrm{Tr}(\boldsymbol{q}_\lambda \, \mathrm{d}\boldsymbol{x}_{\pi_\lambda}). \tag{1.45}$$

Again, we can rearrange the summation using the identity (1.16), which gives

$$\Theta = \tfrac{1}{2} \sum_{\lambda \in \Gamma_+} \mathrm{Tr}(\boldsymbol{q}_\lambda \, \mathrm{d}\boldsymbol{x}_{\pi_\lambda}) + \tfrac{1}{2} \sum_{\lambda \in \Gamma_-} \mathrm{Tr}(\boldsymbol{q}_\lambda \, \mathrm{d}\boldsymbol{x}_{\pi_\lambda}) + \tfrac{1}{2} \sum_{\eta \in \Gamma_{-\infty}} \mathrm{Tr}(\boldsymbol{q}_\eta \, \mathrm{d}\boldsymbol{x}_{\pi_\eta}). \tag{1.46}$$

Then we replace the index λ in the second sum by $-\lambda$, so that this also becomes a sum over Γ_+. And finally we insert the definition of the relative position vectors $\boldsymbol{z}_\lambda$, the energies M_η, and the clocks T_η. The result is

$$\Theta = \tfrac{1}{2} \sum_{\lambda \in \Gamma_+} \mathrm{Tr}(\boldsymbol{q}_\lambda \, \mathrm{d}\boldsymbol{z}_\lambda) - \sum_{\eta \in \Gamma_\infty} M_\eta \, \mathrm{d}T_\eta. \tag{1.47}$$

Up to a total derivative, which can always be added to the symplectic potential, this is equal to (1.35) on the kinematical constraint surface $\mathcal{S}_\Gamma \subset \mathcal{Q}_\Gamma$, because there we have $L_\eta = 0$.

Mass shell constraints

In addition to the kinematical constraints, we also have to impose the *dynamical*, or mass shell constraints, providing the actual time evolution with respect to the ADM time t. They are still given by (1.2), and the Hamiltonian is a general linear combination thereof,

$$\mathcal{H} = \sum_{\pi} \zeta_\pi \, \mathcal{C}_\pi, \qquad \mathcal{C}_\pi = \tfrac{1}{4}\mathrm{Tr}(\boldsymbol{p}_\pi^2) + \tfrac{1}{2} m_\pi^2 \approx 0. \tag{1.48}$$

The subspace $\mathcal{P}_\Gamma \subset \mathcal{S}_\Gamma$ defined by the mass shell constraints is called the *physical phase space*.

To derive the time evolution equations, we have to express the momenta $\boldsymbol{p}_\pi$ as functions of the relative momenta $\boldsymbol{q}_\lambda$ for $\lambda \in \Gamma_0$, and the energies M_η for $\eta \in \Gamma_\infty$. Consequently, the mass shell constraints have non-vanishing brackets with the relative position vectors $\boldsymbol{z}_\lambda$ for $\lambda \in \Gamma_0$, and the clocks T_η for $\eta \in \Gamma_\infty$. A straightforward calculation yields the following brackets of a momentum vector $\boldsymbol{p}_\pi$ with a relative position vector $\boldsymbol{z}_\lambda$,

$$\{p_\pi^a, z_\lambda^b\} = \sum_{\kappa \in \Gamma_\pi} \{q_\kappa^a, z_\lambda^b\} = \begin{cases} \eta^{ab} & \text{if } \lambda \in \Gamma_\pi, \\ -\eta^{ab} & \text{if } \lambda \in \Gamma_{-\pi}, \\ 0 & \text{otherwise.} \end{cases} \tag{1.49}$$

And for the clocks we finds

$$\{p_\pi^a, T_\eta\} = \sum_{\kappa \in \Gamma_\pi} \{q_\kappa^a, T_\eta\} = -\sum_{\kappa \in \Gamma_\pi \cap \Gamma_{-\infty}} \{M_\kappa, T_\eta\} \, \eta^{a0} = \begin{cases} \eta^{a0} & \text{if } \eta \in \Gamma_{-\pi}, \\ 0 & \text{otherwise.} \end{cases} \tag{1.50}$$

Finally, the brackets with the mass shell constraints are

$$\{\mathcal{C}_\pi, \boldsymbol{z}_\lambda\} = \begin{cases} \boldsymbol{p}_\pi & \text{if } \pi = \pi_\lambda, \\ -\boldsymbol{p}_\pi & \text{if } \pi = \pi_{-\lambda}, \\ 0 & \text{otherwise,} \end{cases} \qquad \{\mathcal{C}_\pi, T_\eta\} = \begin{cases} p_\pi^0 & \text{if } \pi = \pi_{-\eta} \\ 0 & \text{otherwise.} \end{cases} \tag{1.51}$$

Using all this, we can easily derive the following brackets with $\mathcal{H}$, providing the time evolution of the link variables with respect to the ADM time t,

$$\dot{\boldsymbol{z}}_\lambda = \{\mathcal{H}, \boldsymbol{z}_\lambda\} = \zeta_{\pi_\lambda} \, \boldsymbol{p}_{\pi_\lambda} - \zeta_{\pi_{-\lambda}} \, \boldsymbol{p}_{\pi_{-\lambda}}, \qquad \dot{T}_\eta = \{\mathcal{H}, T_\eta\} = \zeta_{\pi_{-\eta}} \, p_{\pi_{-\eta}}^0. \tag{1.52}$$

This is exactly how we expect the relative position vectors $\boldsymbol{z}_\lambda$ and the clocks T_η to behave when the particles are moving along their world lines. We should also note that this is consistent with the definitions (1.21) of $\boldsymbol{z}_\lambda$ and (1.23) of T_η, and the original time evolution equations (1.4).

Symmetries

Let us also consider the rigid symmetries of the system at the level of the extended phase space $\mathcal{Q}$. We have seen that the charges are given by

$$M = \sum_{\eta \in \Gamma_\infty} M_\eta, \qquad S = \sum_{\lambda \in \Gamma_+} L_\lambda, \tag{1.53}$$

where M_η are the energies assigned to the external links, and L_λ are the angular momenta of the internal links, which were given by (1.30). The rigid symmetry generated by M is very simple. The only non-vanishing brackets are

$$\{M, T_\eta\} = -1 \quad \Rightarrow \quad T_\eta \mapsto T_\eta - \Delta T, \tag{1.54}$$

where ΔT is some real parameter. The symmetry only acts on the clocks, which are all turned backwards by the same amount. In figure 1, this corresponds to a translation of the ADM surface backwards in time, or to a forward time translation of the reference frame with respect to the particles. The total energy M is thus the generator of the absolute time evolution in the centre of mass frame. This must be distinguished from the ADM time evolution with respect to the coordinate time t, which is generated by the weakly vanishing, unphysical Hamiltonian $\mathcal{H}$.

To find the symmetry generated by S, let us first evaluate the brackets of the link variables with L_λ. A straightforward calculation shows that the only non-vanishing brackets are

$$\{L_\lambda, \boldsymbol{z}_{\pm\lambda}\} = \tfrac{1}{2}[\boldsymbol{\gamma}_0, \boldsymbol{z}_{\pm\lambda}], \qquad \{L_\lambda, \boldsymbol{q}_{\pm\lambda}\} = \tfrac{1}{2}[\boldsymbol{\gamma}_0, \boldsymbol{q}_{\pm\lambda}], \qquad \lambda \in \Gamma_0. \tag{1.55}$$

This is the generator of a counter clockwise rotation about the $\boldsymbol{\gamma}_0$-axis, which acts on the vectors associated with the links λ and $-\lambda$. If we sum over all internal links $\lambda \in \Gamma_+$, then the transformation acts in the same way on all internal links,

$$\{S, \boldsymbol{z}_\lambda\} = \tfrac{1}{2}[\boldsymbol{\gamma}_0, \boldsymbol{z}_\lambda], \qquad \{S, \boldsymbol{q}_\lambda\} = \tfrac{1}{2}[\boldsymbol{\gamma}_0, \boldsymbol{q}_\lambda], \qquad \lambda \in \Gamma_0. \tag{1.56}$$

The resulting transformation is a rotation (A.11) about the $\boldsymbol{\gamma}_0$-axis,

$$\boldsymbol{q}_\lambda \mapsto \mathrm{e}^{\Delta\phi\boldsymbol{\gamma}_0/2}\boldsymbol{q}_\lambda\, \mathrm{e}^{-\Delta\phi\boldsymbol{\gamma}_0/2}, \qquad \boldsymbol{z}_\lambda \mapsto \mathrm{e}^{\Delta\phi\boldsymbol{\gamma}_0/2}\boldsymbol{z}_\lambda\, \mathrm{e}^{-\Delta\phi\boldsymbol{\gamma}_0/2}, \qquad \lambda \in \Gamma_0, \tag{1.57}$$

where $\Delta\phi$ is the angle of rotation. But now, we have the following problem. Consider the action of this transformation on the polygons in figure 2. All internal links are rotated by an angle $\Delta\phi$ in counter clockwise direction. But the directions of the external links, specified by the unit vectors $\boldsymbol{z}_{\pm\eta}$, are unchanged. As a consequence, there is a certain restriction on the parameter $\Delta\phi$, because for too large angles the non-compact polygons are twisted in such a way that they can no longer be realized as spacelike surfaces.

This can be avoided as follows. We redefine the total angular momentum, including also the angular momenta L_η of the external links,

$$J = \sum_{\lambda\in\Gamma_+} L_\lambda + \sum_{\eta\in\Gamma_\infty} L_\eta. \tag{1.58}$$

Due to the constraints (1.40), the charges S and J are in fact weakly equal, thus $S \approx J$. The associated rigid symmetries are therefore equal up to a kinematical gauge symmetry. For J, we have the additional non-vanishing brackets

$$\{J, \phi_\eta\} = 1 \quad \Rightarrow \quad \{J, \boldsymbol{z}_{\pm\eta}\} = \tfrac{1}{2}[\boldsymbol{\gamma}_0, \boldsymbol{z}_{\pm\eta}], \qquad \eta \in \Gamma_\infty. \tag{1.59}$$

The second equation follows from the definition (1.34) of the unit vectors $\boldsymbol{z}_{\pm\eta}$. Hence, if we replace S by J, then the transformation (1.57) also applies to the external links. For the relative momenta $\boldsymbol{q}_{\pm\eta}$ this is trivial, because they are proportional to $\boldsymbol{\gamma}_0$ and therefore invariant, and for the unit vectors $\boldsymbol{z}_{\pm\eta}$ it follows from (1.59). The charge J generates a rigid rotation of the ADM surface with respect to the reference frame, without deforming it. For such a transformation, the parameter $\Delta\phi$ can be arbitrarily large, and for $\Delta\phi = 2\pi$ we get a full rotation and thus the identity. On the extended phase space $\mathcal{Q}$, it is therefore more natural to call J the total angular momentum, but for all physical states its value is of course equal to S.

Global aspects

There are some global features of the extended phase space which we neglected so far. For example, until now we have fixed the graph Γ. But it is not possible to use the same graph for every possible configuration of the particles. We therefore have to take into account that the extended phase space consists of finitely many disconnected components. The same, of course, applies to the kinematical and the physical subspace,

$$\mathcal{Q} = \bigcup_{\Gamma} \mathcal{Q}_{\Gamma}, \qquad \mathcal{S} = \bigcup_{\Gamma} \mathcal{S}_{\Gamma}, \qquad \mathcal{P} = \bigcup_{\Gamma} \mathcal{P}_{\Gamma}. \tag{1.60}$$

On the other hand, the original phase space of the particles was a connected manifold. We previously argued that this phase space is equal to the quotient space $\mathcal{S}_{\Gamma}/\!\sim$. But this identity only holds locally. Each components of the quotient space only covers a certain region of the original phase space, namely that region where the graph Γ defines a proper triangulation of the ADM surface.

So, what we actually have is an atlas of finitely many charts covering the original phase space of the particles. An alternative point of view is to consider the transitions between different triangulations as *large gauge symmetries*. The quotient space $\mathcal{S}/\!\sim$, where the large gauge symmetries are also divided out, is then globally equal to the original phase space. We shall look at this global structure of the phase space more closely in section 2, where it is schematically indicated in figure 8.

It is also useful to keep in mind that following hierarchy of phase spaces, which is very similar to the Hamiltonian formulation of general relativity,

$$\begin{array}{ccccc} \mathcal{Q} & \supset & \mathcal{S} & \supset & \mathcal{P} \\ & & \downarrow & & \downarrow \\ & & \mathcal{S}/\!\sim & & \mathcal{P}/\!\sim. \end{array} \tag{1.61}$$

The starting point is the extended phase space $\mathcal{Q}$. The kinematical constraints (1.38–1.40) define the kinematical subspace $\mathcal{S}$, and the mass shell constraints (1.48) define the physical phase space $\mathcal{P}$. At each level, the gauge symmetries associated with the respective constraints can be divided out. The quotient space $\mathcal{S}/\!\sim$ is the original phase space, where the redundancies are removed. And the quotient space $\mathcal{P}/\!\sim$ is the reduced phase space, which is the set of all classical trajectories, that is all physically inequivalent solutions to the equations of motion.

Finally, there is another somewhat marginal point, which also has to do with the global structure of the phase space. The kinematical constraints alone are not sufficient to ensure that the geometry of the ADM surface in figure 1 is well defined. For the polygons in figure 2 to be well defined, the sequences of edges defined by the vectors $\boldsymbol{z}_\lambda$ must be boundaries of spacelike surfaces. A necessary condition is of course that all vectors $\boldsymbol{z}_\lambda$ are spacelike. But there are more consistency conditions. For a compact triangle, for example, the vector product of two successive edges must be negative timelike, for the polygon surface to be spacelike with the correct orientation. For higher polygons and also for non-compact polygons there are more complicated conditions.

However, they can all be written as inequalities involving the vectors $\boldsymbol{z}_\lambda$, and this is all we need to know. They can be treated at the same level as the positive energy conditions, which we have

to impose together with the mass shell constraints. They pick out an open subset of the actual constraint surface, which is the physically accessible subset. The kinematical phase space $\mathcal{S}_\Gamma$ is this subset of the constraint surface, similar to the physical phase space $\mathcal{P}_\Gamma$, which is that part of the constraint surface where the positive energy conditions are satisfied. This further restriction has no influence on the local structure of the phase space, for example number of gauge degrees of freedom. But it imposes some restriction on the parameters of the gauge transformations, as otherwise we drop out of the kinematical phase space.

For example, we cannot rotate the external links arbitrarily far, thus there is a restriction on the parameters ϵ_η in (1.31), as otherwise the external polygons are no longer spacelike. This was also the reason why the transformation generated by S as a conserved charge is not well defined for all parameters. And there is also a restriction on the multipliers ζ_π for the mass shell constraints. We cannot shift a particle along its world line arbitrarily far, as otherwise the polygons attached to this particle are no longer spacelike. Of course, this already follows from the original restriction (1.5), so this restriction has already been present before we introduced the link variables and the extended phase space.

2 Spacetime geometry

The gravitational field of a massive point particle in three dimensional Einstein gravity is a cone with a deficit angle of $8\pi Gm$, where $0 < m < M_{\mathrm{Pl}}/4$ is the mass of the particle and G is Newton's constant, which is the inverse of the Planck mass $M_{\mathrm{Pl}} = 1/G$. In the neighbourhood of the world line, a cylindrical coordinate system (t, r, φ) can be introduced, so that the metric becomes

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + (1 - 4Gm)^2\, r^2\, \mathrm{d}\varphi^2. \tag{2.1}$$

This metric is locally flat for $r > 0$, and there is a conical singularity on the world line at $r = 0$. A spacetime containing n particles can in principle be covered by an atlas of n such conical coordinate systems, and the relative motion of the particles can be read off from the respective transition functions.

And alternative and more appropriate way to introduce coordinates is first to consider the spacetime between the particles. Let us remove the world lines from the spacetime. It is then a locally flat manifold, which is not simply connected. If a spacetime vector is transported, say, along a path that winds around a world line in clockwise direction, then we pick up a Lorentz rotation, which is called the *holonomy* of the particle. It does not depend on the precise path, and is therefore a quantity that can be assigned to the particle. If the deficit angle of the conical singularity is $8\pi Gm$, then the holonomy is a rotation by $8\pi Gm$ about some timelike axis, which is parallel to the world line.

From the holonomy we can read off both the mass and the direction of the motion of the particle. It can be regarded as a generalized momentum [13]. So, we have in this picture a well defined notion of a momentum. Massless particles can also be included. For a massless particle, the world line is lightlike, and the holonomy is a null rotation, whereas a conical coordinate system like (2.1) does not exist. It is also possible to introduce relative position coordinates of the particles, very similar to the free particle coordinates in the previous section. So let us make all this a little bit more explicit.

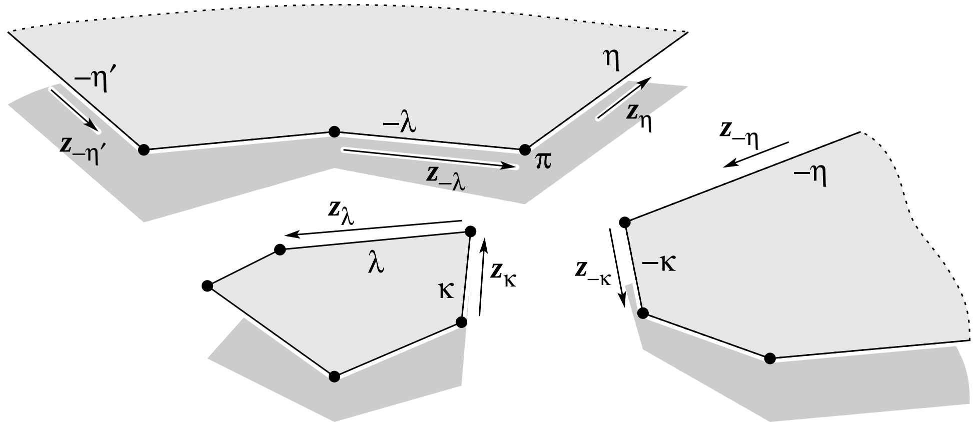

Figure 3: The ADM surface is a conical surface with n tips. It is divided into a collection of simply connected polygons. The polygons are embedded into an auxiliary Minkowski space, which provides an atlas of local coordinate charts. The edges $\pm\lambda$ of two adjacent polygons $\Delta_{\pm\lambda}$ are mapped onto each other by some Lorentz rotation. This ensures that the polygons can be glued together, forming a spacelike surface, which is embedded into a locally flat spacetime with n conical singularities.

Minkowski coordinates

Suppose that the spacetime is foliated by a family of spacelike slices, labeled by an ADM time coordinate t. Then we would like to know, for example, what the relative position of two particles is at a moment of time t. Consider the corresponding ADM surface, and some neighbourhood of this surface in spacetime. If the world lines are excluded, then this is a spacelike surface embedded into a locally flat spacetime. In this locally flat spacetime, we can introduce an atlas of *Minkowski charts*. This can be done as follows. First, we triangulate the ADM surface in the same way as before. We introduce a graph $\mathit{\Gamma}$, so that the links $\lambda \in \mathit{\Gamma}$ are spacetime geodesics, and the ADM surface is divided into a collection $\mathit{\Pi}$ of simply connected polygons $\Delta \in \mathit{\Pi}$.

For simplicity, let us assume that the topology of space is that of $\mathbb{R}^2$. Possible generalizations will be discussed later. Then, in the neighbourhood of each polygon, we introduce a Minkowski coordinate system. This is possible because the spacetime is locally flat, and the neighbourhood of the polygon in spacetime is simply connected. The polygons become spacelike surfaces, which are embedded into some auxiliary Minkowski space, as indicated in figure 3. Locally, this embedding Minkowski space can be identified with the spacetime. Each edge $\lambda \in \mathit{\Gamma}_\Delta$ of a polygon Δ represents a geodesic connecting two particles. It becomes a straight line in Minkowski space.

To this we can assign a vector $\boldsymbol{z}_\lambda \in \mathfrak{sl}(2)$, in the same way as before. It represents the relative position of the particles π_λ and $\pi_{-\lambda}$ in spacetime. Vice versa, the vectors $\boldsymbol{z}_\lambda$ determine the geometry of the individual polygons in figure 3. There are only the following modifications, as compared to polygons in figure 2. The vectors $\boldsymbol{z}_\lambda$ now refer to local coordinates on the spacetime, and not to a fixed reference frame. We are allowed to perform coordinate transformations. On

the embedded polygons in figure 3, a coordinate transformation acts as a Lorentz rotation and a translation, and the vectors $\boldsymbol{z}_\lambda$ are also Lorentz rotated.

Moreover, for each pair of edges λ and $-\lambda$, the vectors $\boldsymbol{z}_\lambda$ and $\boldsymbol{z}_{-\lambda}$ still represent the relative position of the same two particles. But now they refer to different local coordinates systems. The vector $\boldsymbol{z}_\lambda$ is defined in the chart that contains the polygon Δ_λ, and the vector $\boldsymbol{z}_{-\lambda}$ refers to the chart containing the polygon $\Delta_{-\lambda}$. There is an overlap region of the two charts, which contains the links λ and $-\lambda$. The transition function between the two charts must be an isometry of Minkowski space, thus a Poincaré transformation. Let us only consider the rotational component of this transition function. It is represented by an element $\boldsymbol{g}_\lambda \in \mathsf{SL}(2)$ of the Lorentz group.

In figure 3, this means that the edge $-\lambda$ of the polygon Δ_λ is mapped onto the edge λ of the polygon Δ_λ by a Lorentz rotation given by $\boldsymbol{g}_\lambda$. Of course, $\boldsymbol{g}_{-\lambda}$ is then the inverse transformation. We have the following relation between the relative position vectors and the transition functions,

$$\boldsymbol{g}_{-\lambda} = \boldsymbol{g}_\lambda^{-1}, \qquad \boldsymbol{z}_{-\lambda} = -\boldsymbol{g}_\lambda \, \boldsymbol{z}_\lambda \, \boldsymbol{g}_\lambda^{-1}, \qquad \lambda \in \mathit{\Gamma}. \tag{2.2}$$

This relation also applies to the external edges in figure 3. The spacelike unit vectors $\boldsymbol{z}_{\pm\eta}$ define the directions of the external edges in the embedding Minkowski space. And there is also a transition function associated with these links, defining the relation between two adjacent non-compact charts.

Now, suppose we are given the vectors $\boldsymbol{z}_\lambda$ and the transition functions $\boldsymbol{g}_\lambda$ for all $\lambda \in \mathit{\Gamma}$. The geometry of the polygons in figure 3 is then determined up to smooth deformation of the surfaces in the interior. The transition functions tell us how these polygons are to be glued together. This determines the geometry of the ADM surface, and the way it is embedded into the spacetime. So, we conclude that the relative position vectors $\boldsymbol{z}_\lambda$ and the transition functions $\boldsymbol{z}_\lambda$ specify the geometry of space at a moment of time, up to some physically redundant gauge symmetries. The deformations are in fact gauge symmetries of general relativity, namely spacetime diffeomorphisms acting on the foliation.

It is useful to think about the polygons in figure 3 as a *deformation* of the polygons in figure 2, with Newton's constant being the deformation parameter. We can take a limit $G \to 0$, where the gravitational interaction is switched off, if we relate the transition functions $\boldsymbol{g}_\lambda$ in a certain way to the relative momentum vectors $\boldsymbol{q}_\lambda$ of the free particles. Let us define

$$\boldsymbol{g}_\lambda = \mathrm{e}^{4\pi G \boldsymbol{q}_\lambda} = \mathbf{1} + 4\pi G \, \boldsymbol{q}_\lambda + O(G)^2, \qquad \lambda \in \mathit{\Gamma}. \tag{2.3}$$

Newton's constant has to show up in this relation, because the vector $\boldsymbol{q}_\lambda$ has the physical dimension of a momentum, whereas $\boldsymbol{g}_\lambda$ is dimensionless. The numerical factor will be explained in a moment. If we insert this into (2.2), and take the limit $G \to 0$, then in the leading order in G we recover the relations (1.33). Moreover, for all transition functions we have $\boldsymbol{g}_\lambda \to \mathbf{1}$, so that the individual Minkowski charts can be unified, providing one global chart. So, in this limit we recover the triangulated ADM surface of the free particles, which is globally embedded into a flat Minkowski space.

It is also useful to look at the way the link variables transform under coordinate transformation in the Minkowski charts. In each chart, we can act on the coordinates with some Poincaré transformation. On the corresponding polygon in figure 3, the transformation acts as a Lorentz rotation and a translation. Since none of the link variables refers to the absolute position of the

polygons in the embedding Minkowski space, we can again on the Lorentz rotations and ignore the translations. So, let $\boldsymbol{h}_\Delta \in \mathsf{SL}(2)$ be the Lorentz rotation acting on the chart containing the polygon Δ. Then, one can easily verify that the link variables transform as

$$\boldsymbol{g}_\lambda \mapsto \boldsymbol{h}_{\Delta_{-\lambda}} \, \boldsymbol{g}_\lambda \, \boldsymbol{h}_{\Delta_\lambda}^{-1}, \qquad \boldsymbol{z}_\lambda \mapsto \boldsymbol{h}_{\Delta_\lambda} \, \boldsymbol{z}_\lambda \, \boldsymbol{h}_{\Delta_\lambda}^{-1}. \tag{2.4}$$

Note that the transition function $\boldsymbol{g}_\lambda$ sees the transformations acting on both polygons Δ_λ and $\Delta_{-\lambda}$, whereas the vector $\boldsymbol{z}_\lambda$ only refers to the polygon Δ_λ and transforms accordingly.

To see what this coordinate transformation looks like when the gravitational interaction is switched off, we replace the parameter $\boldsymbol{h}_\Delta$ by a vector $\boldsymbol{\chi}_\Delta$, so that

$$\boldsymbol{h}_\Delta = \mathrm{e}^{4\pi G \boldsymbol{\chi}_\Delta} = \mathbf{1} + 4\pi G \, \boldsymbol{\chi}_\lambda + O(G)^2, \qquad \Delta \in \mathit{\Pi}. \tag{2.5}$$

In the limit $G \to 0$, the Lorentz rotation (2.4) then reduces to the redundancy transformation (1.13) for the relative momentum vectors of the free particles, again in the leading order in G. The coordinate transformations in the Minkowski charts are the redundancies of the interacting particle system, analogous to the redundancies of the relative momenta for the free particles.

Holonomies

When the polygons are glued together, then at each vertex of the triangulation a conical singularity arises. Consider a spacetime vector which is transported once around a particle π, in clockwise direction. As indicated in figure 4, the vector is defined in some polygon Δ adjacent to the particle π. The path winds once around the particle. Whenever it hits a link $-\lambda \in \mathit{\Gamma}_{-\pi}$ beginning at π, then it continues from the link $\lambda \in \mathit{\Gamma}_\pi$ ending at π, in the next polygon. The sequence of in which the links are crossed is given by the cyclic ordering of the set $\mathit{\Gamma}_\pi$. When the vector is transferred from the link $-\lambda$ to the link λ, hence from one chart to another, then we have to act on it with the transition function $\boldsymbol{g}_\lambda$.

The holonomy $\boldsymbol{u}_\pi \in \mathsf{SL}(2)$ of the particle π is the product of the transition functions $\boldsymbol{g}_\lambda$ for all $\lambda \in \mathit{\Gamma}_\pi$. The ordering of the factors is defined by the cyclic ordering of the set $\mathit{\Gamma}_\pi$. Which factor is the first depends on the polygon Δ in which the path begins and ends. Let us introduce the following notation. If π is a particle, and Δ is a polygon adjacent to it, then $\mathit{\Gamma}_{\pi,\Delta} = \{\lambda', \dots, \lambda\}$ is the ordered set of all links ending π, so that the first and last elements $\lambda, \lambda' \in \mathit{\Gamma}_\pi$ are those successive links ending at π, which enclose the polygon $\Delta = \Delta_\lambda = \Delta_{-\lambda'}$. We say that the cyclic ordering of the set $\mathit{\Gamma}_\pi$ is *broken* at the polygon Δ.

With this notation, the holonomy of the particle π, evaluated in the polygon Δ, becomes

$$\boldsymbol{u}_{\pi,\Delta} = \prod_{\lambda \in \mathit{\Gamma}_{\pi,\Delta}} \boldsymbol{g}_\lambda. \tag{2.6}$$

There are as many holonomies $\boldsymbol{u}_{\pi,\Delta}$ of the particle π as there are polygons Δ adjacent to the vertex π. They are, however, just representatives of the same physical object in different coordinate charts. To see this, consider the behaviour of the holonomy under Lorentz rotations (2.4) of the Minkowski charts,

$$\boldsymbol{u}_{\pi,\Delta} \mapsto \boldsymbol{h}_\Delta \, \boldsymbol{u}_{\pi,\Delta} \, \boldsymbol{h}_\Delta^{-1}. \tag{2.7}$$

Hence, $\boldsymbol{u}_{\pi,\Delta}$ transforms according to the Lorentz rotation acting on the polygon Δ. Moreover, consider the representatives of the holonomy $\boldsymbol{u}_{\pi,\Delta_\lambda}$ and $\boldsymbol{u}_{\pi,\Delta_{-\lambda}}$ in the two polygons, sharing a

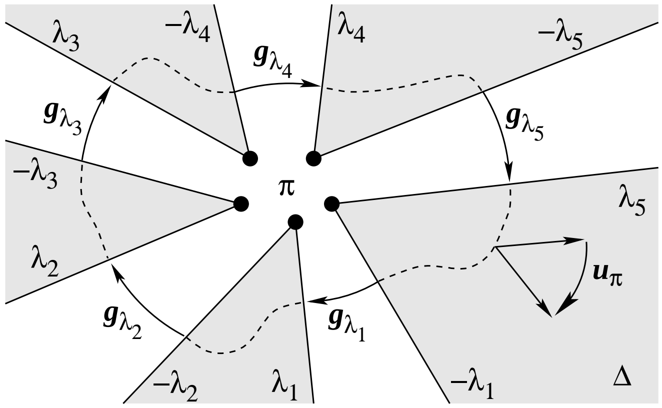

Figure 4: There is a conical singularity at each vertex of the triangulation. Its holonomy $\boldsymbol{u}_\pi$, evaluated in the polygon Δ, is the product of the transitions functions $\boldsymbol{g}_\lambda$ for $\lambda \in \mathit{\Gamma}_{\pi,\Delta}$. In the given example, we have $\mathit{\Gamma}_{\pi,\Delta} = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\}$, because $-\lambda_1$ and λ_5 are the edges of the polygon Δ.

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common boundary $\lambda \in \mathit{\Gamma}_\pi$ and $-\lambda \in \mathit{\Gamma}_{-\pi}$. For these two representatives, the definition (2.6) differs by the position of the factor $\boldsymbol{g}_\lambda$. For $\Delta = \Delta_\lambda$ it is the last factor, whereas for $\Delta = \Delta_{-\lambda}$ it is the first factor in the product. This implies

$$\boldsymbol{u}_{\pi,\Delta_\lambda} = \boldsymbol{g}_\lambda^{-1} \, \boldsymbol{u}_{\pi,\Delta_{-\lambda}} \, \boldsymbol{g}_\lambda. \tag{2.8}$$

But this is just the statement that $\boldsymbol{u}_{\pi,\Delta_{-\lambda}}$ and $\boldsymbol{u}_{\pi,\Delta_\lambda}$ represent the same physical object in two different coordinate charts, where $\boldsymbol{g}_\lambda$ is the transition function between the two charts. Note that if there is only one link λ attached to the particle π, then there is also only one polygon Δ, and the relation (2.8) is just a trivial identity, because $\boldsymbol{u}_{\pi,\Delta} = \boldsymbol{g}_\lambda$ and $\Delta = \Delta_\lambda = \Delta_{-\lambda}$.

According to the arguments given in the beginning, we can now read off the mass of the particle and the direction of motion from the holonomy. The direction of motion is the axis of the Lorentz rotation represented by the holonomy. This axis can be defined by *projecting* the group element $\boldsymbol{u}_\pi$ onto a vector $\boldsymbol{p}_\pi$, which we call the *momentum vector* of the particle. According to (A.12), the projection $\mathsf{SL}(2) \to \mathfrak{sl}(2)$ is defined by expanding a group element in terms of the unit and the gamma matrix, and then dropping the term proportional to the unit matrix. To give the momentum vector the correct physical dimension, we introduce Newton's constant once again, and define

$$\boldsymbol{u}_{\pi,\Delta} = u_\pi \, \mathbf{1} + 4\pi G \, p^a_{\pi,\Delta} \, \boldsymbol{\gamma}_a, \qquad \boldsymbol{p}_{\pi,\Delta} = p^a_{\pi,\Delta} \, \boldsymbol{\gamma}_a. \tag{2.9}$$

By definition, the momentum vector commutes, as a matrix, with the holonomy, and therefore defines the axis of the Lorentz rotation in Minkowski space. It transforms in the same way (2.7) under coordinate transformations, and between the representatives in the different charts we also have the relation (2.8), with $\boldsymbol{u}_\pi$ replaced by $\boldsymbol{p}_\pi$.

In the limit $G \to 0$, and with (2.3) inserted, the product in (2.6) can be replaced by a sum. In the leading order in G, we recover the definition (1.12) of the momentum vector of the free

particle, which is the sum of the relative momentum vectors. To explain the numerical factors appearing together with Newton's constant, let us derive the relation between the holonomy and the mass of the particle. For the mass to be m_π, the holonomy must be a rotation by $8\pi G m_\pi$ in clockwise direction about some timelike axis. All such element of the group $\mathsf{SL}(2)$ belong to the conjugacy class of $\mathrm{e}^{4\pi G m_\pi \gamma_0}$, which represents a clockwise rotation about the $\boldsymbol{\gamma}_0$-axis.

This conjugacy class of $\mathsf{SL}(2)$ is specified by the following *mass shell* and *positive energy* condition. The scalar u_π in (2.9), which is half the trace of the holonomy, must be equal to $\cos(4\pi G m_\pi)$, and the $\boldsymbol{\gamma}_0$-component of the momentum $\boldsymbol{p}_\pi$ must be positive to fix the direction of the rotation. Hence,

$$u_\pi = \cos(4\pi G m_\pi), \qquad p_\pi^0 = \tfrac{1}{2}\mathrm{Tr}(\boldsymbol{u}_\pi \boldsymbol{\gamma}^0) > 0. \tag{2.10}$$

Note that the trace of the holonomy is a scalar. It is the same in every polygon, because the trace is invariant under cyclic permutations of the factors,

$$u_\pi = \tfrac{1}{2}\mathrm{Tr}(\boldsymbol{u}_\pi) = \tfrac{1}{2}\mathrm{Tr}\Big(\prod_{\lambda \in \Gamma_\pi} \boldsymbol{g}_\lambda\Big), \tag{2.11}$$

and it transform trivially under coordinate transformations. The mass shell condition is therefore a well defined, coordinate independent equation. Actually, there are some subtleties regarding the global structure of the group manifold $\mathsf{SL}(2)$. But these can be ignored here, and we can take (2.10) as a condition for the mass of the particle to be m_π. A comprehensive discussion can be found in [13].

It is now also possible to include massless particles. For $m_\pi = 0$, the holonomy becomes a null rotation and the momentum vector is a lightlike vector. Thus a massless particle moves with the velocity of light. On the other hand, there is an upper bound on the mass, because the deficit angle of a conical singularity must be smaller than 2π. The upper bound is $M_{\mathrm{Pl}}/4 = 1/4G$. This is also the value where the cosine takes its minimum -1, and the holonomy becomes a full rotation by 2π. The allowed range of the mass parameters is

$$0 \le m_\pi < M_{\mathrm{Pl}}/4 = 1/4G. \tag{2.12}$$

And finally, to explain the numerical factors, let us consider the limit $G \to 0$ once again. Between the trace of the holonomy and the momentum vector we have the relation (A.13),

$$u_\pi^2 = 8\pi^2 G^2\, \mathrm{Tr}(\boldsymbol{p}_\pi^2) + 1. \tag{2.13}$$

Therefore, the mass shell condition implies

$$\tfrac{1}{2}\mathrm{Tr}(\boldsymbol{p}_\pi^2) = -\frac{\sin^2(4\pi G m_k)}{16\pi^2 G^2} \tag{2.14}$$

In the limit $G \to 0$, this obviously becomes the free particle mass shell condition (1.3).

The conical infinity

There is one feature of the free particle system which has so far no counterpart for the interacting system. This is the *reference frame*, defined by the embedding Minkowski space. The

Minkowski space in figure 3 is now only an auxiliary space, providing local coordinates on the spacetime manifold. Since we are free to perform coordinate transformations, rotating the individual polygons, this Minkowski space does not provide a well defined reference frame. On the other hand, a reference frame is needed if we want to set up a proper Hamiltonian formulation in general relativity [19].

To define a reference frame, we have to look at the asymptotic structure of the spacetime at spatial infinity. Let us assume that, far away from the particles, the spacetime looks like the gravitational field of a single particle. We'll see in section 4, that this can be formulated as a kind of asymptotical flatness condition, hence a fall off condition imposed on the metric at infinity. If this is the case, then an observer at infinity effectively sees a *fictitious* centre of mass particle, and the rest frame of this particle defines the centre of mass frame of the universe. We may therefore identify the reference frame with the centre of mass frame, in the same way as we did this in the previous section.

The gravitational field of the fictitious centre of mass particle is a cone like (2.1). However, we have to allow this particle to have a variable mass M, which represents the total energy of the universe. And it also receives a spin S, which represents the total angular momentum of the universe. It is then possible to introduce a *conical coordinate* system (T, R, ϕ). It covers a certain region of the spacetime, far away from the particle, so that the metric becomes that of a *spinning cone* [5],

$$\mathrm{d}s^2 = -(\mathrm{d}T + 4GS\,\mathrm{d}\phi)^2 + \mathrm{d}R^2 + (1-4GM)^2\,R^2\,\mathrm{d}\phi^2. \tag{2.15}$$

The spinning cone has a *deficit angle* of $8\pi GM$, and a *time offset* of $8\pi GS$. There is an upper bound on M which is the same as (2.12). For convenience, let us also assume that M is positive. This is actually not needed, but one can show that the positive energy conditions for the particles imply that also the total energy M is positive. Hence

$$0 < M < M_{\mathrm{Pl}}/4 = 1/4G. \tag{2.16}$$

The conical coordinate system (T, R, ϕ) can then be regarded as a fixed reference frame. It replaces the Minkowski frame of the free particle system. The radial coordinate R defines the distance from the fictitious world line of the centre of mass, the time coordinate T defines the absolute time in the reference frame, and the angular coordinate ϕ, which has a period of 2π, defines the angular orientation of the reference frame. If we take the limit $G \to 0$, with M and S fixed, the spinning cone becomes a flat Minkowski space, and the fictitious world line of the centre of mass is the $\boldsymbol{\gamma}_0$-axis.

So, the reference frame is also a deformation of its free particle counterpart. There is only the following technical difference. The reference frame does not provide a *global* coordinate system, which covers the whole spacetime. It only provides another local chart, which covers a *neighbourhood of infinity*. This is the region outside a cylinder surrounding all the world lines. To define the absolute positions of the particles with respect to this reference frame, we cannot just read off their coordinates. The particles are not inside the chart where (2.15) applies. We have to define them indirectly. At each moment of time, we have to tell how the ADM surface is embedded into the spinning cone at spatial infinity.

Given the geometry of the ADM surface, we have to fix two additional parameters to define its embedding into the spinning cone, because there are two Killing symmetries of the metric

(2.15). These are the time translations $T \mapsto T - \Delta T$ and the spatial rotations $\phi \mapsto \phi + \Delta\phi$. Hence we have the same rigid symmetries of the centre of mass frame as before. The symmetries are the possible rotations and translations of the reference frame with respect to the rest of the universe [19]. To fix this freedom, we have to assign some additional variables to the external links, telling us how they behave when the ADM surface is embedded into the reference frame.

Since we are still free to deform the ADM surface smoothly, without affecting the locations of the particles, let us impose the following restriction on the external links. As for the free particles, we require them to be *spatial half lines*. A spatial half line in the spinning cone is a spacelike geodesic extending to infinity, which is orthogonal to the fictitious axis at $R = 0$. An alternative definition is to say that a spatial half line is orthogonal to the Killing field of time translations. This is obviously a straightforward generalization of the definition in Minkowski space, where a spatial half line is orthogonal to the $\boldsymbol{\gamma}_0$-axis.

One can then easily show that on every spatial half line the conical coordinates T and ϕ then converge in the limit where R goes to infinity. So, on each external link we have

$$T \to T_\eta, \qquad \phi \to \phi_\eta, \qquad R \to \infty, \qquad \eta \in \mathit{\Gamma}_\infty. \tag{2.17}$$

The physical interpretation of the link variables T_η and ϕ_η is almost the same as before. Let us consider an observer sitting at the far end of the link η, hence at spatial infinity. The variable ϕ_η tells us where this observer is, that is in which *direction*, and the variable T_η represents the absolute time at the location of this observer, hence it defines a *clock*.

On the other hand, given the geometry of the ADM surface, specified by the link variables $\boldsymbol{z}_\lambda$ and $\boldsymbol{g}_\lambda$, and also the variables T_η and ϕ_η, we know how to embed the space manifold into the spinning cone, and thus we have complete information about the state of the interacting particle system at a given moment of time. We know the relative positions of the particles with respect to each other, and the absolute positions with respect to the reference frame.

Consistency conditions

There are also some consistency conditions to be satisfied by the link variables, in analogy to the kinematical constraints for the free particles. First of all, for a polygon Δ in figure 3 to be well defined, the sequence of edges given by the vectors $\boldsymbol{z}_\lambda$ for $\lambda \in \mathit{\Gamma}_\Delta$ must be the boundary of spacelike surface. For each compact polygon we must have

$$\sum_{\lambda \in \mathit{\Gamma}_\Delta} \boldsymbol{z}_\lambda = 0, \qquad \Delta \in \mathit{\Pi}_0. \tag{2.18}$$

Additionally, there are some inequalities to be satisfied, for example all vectors $\boldsymbol{z}_\lambda$ must be spacelike, and the boundaries must be correctly oriented. The precise form of these inequalities is still not important, for the same reasons as before, which we discussed in the very end of section 1.

Then there are also some consistency conditions for the ADM surface to fit into the spinning cone at spatial infinity. It must be possible to embed the non-compact polygons into the spinning cone, so that the edges fit together correctly. We have indicated this, somewhat schematically, in figure 5. Each non-compact polygon $\Delta \in \mathit{\Pi}_\infty$ covers a certain segment of the spinning cone, which is bounded by the external links η and η'. They are spatial half lines with conical

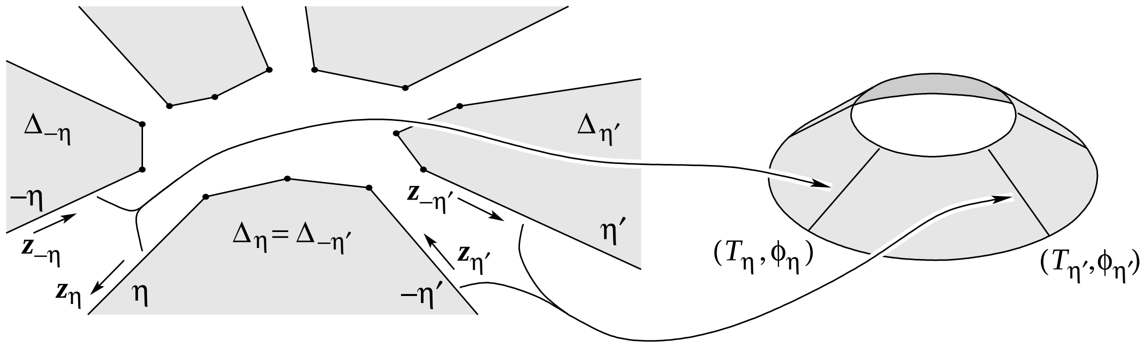

Figure 5: The non-compact polygons are embedded into the spinning cone. The spinning cone on the right defines the reference frame, whereas the Minkowski space on the left only provides local coordinates. The vectors $\boldsymbol{z}_{\pm\eta}$ are related to the conical directions ϕ_η of the links in the spinning cone, in such a way that the average orientation of the Minkowski frame is the same as that of the conical frame, so that effectively the embedding Minkowski space can also be regarded as a reference frame.

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coordinates (2.17). For the polygons to fit in, there must be some relation between the vectors $\boldsymbol{z}_\lambda$, defining the geometry of the polygons in the embedding Minkowski space, and the external links variables T_η and ϕ_η.

To find this relation, let us impose one further restriction. We are still free to perform arbitrary Lorentz rotations in all Minkowski charts, including the non-compact ones. On the other hand, each non-compact chart has some overlap region with the conical chart at infinity. Hence, there is a transition function between the Minkowski coordinates and the conical coordinates. This transition function is a local isometry between the spinning cone and Minkowski space. There is a special class of such isometries. They have the property that the axis of the spinning cone is mapped onto the $\boldsymbol{\gamma}_0$-axis of Minkowski space. The most general such local isometry is given by

$$(T, R, \phi) \quad \mapsto \quad (T + \tau_\Delta(\phi))\, \boldsymbol{\gamma}_0 + R\, \boldsymbol{\gamma}(\phi - \alpha_\Delta(\phi)), \tag{2.19}$$

where α and τ are two linear functions satisfying

$$\tau'_\Delta(\phi) = 4GS \qquad \alpha'_\Delta(\phi) = 4GM. \tag{2.20}$$

Note that this is only a local isometry, because it does not respect the periodicity of ϕ. There is a two parameter family of such local isometries, because we are free to add constants to τ_Δ and α_Δ. Clearly, this corresponds to a time translation and a spatial rotation, either in the spinning cone or in Minkowski space.

Let us choose the Minkowski coordinates in the non-compact charts so that the transition function to the conical coordinates is given by such a local isometry. Hence, for each non-compact polygon $\Delta \in \Pi_\infty$ there exists a pair of linear functions τ_Δ and α_Δ, so that in the overlap region of the respective Minkowski chart with the neighbourhood of infinity the transition

function (2.19) applies. What can we then say about the unit vectors $\boldsymbol{z}_{\pm\eta}$ and the transition functions $\boldsymbol{g}_{\pm\eta}$ associated with the external links?

Consider first the transition functions. Each pair of external links $\pm\eta$ represents a triple overlap region between two Minkowski charts, containing the polygons $\Delta_{\pm\eta}$, and the conical chart, which provides a globally defined chart in a neighbourhood of infinity. There is thus a certain relation which involves the transition functions $\boldsymbol{g}_{\pm\eta}$ between the two Minkowski charts, and the transition functions (2.19) to the conical coordinates. The Minkowski coordinates in the two adjacent charts differ by a time translation, which is given by the difference $\tau_{\Delta_\lambda}(\phi) - \tau_{\Delta_{-\lambda}}(\phi)$, and a spatial rotation, which is given by the difference $\tau_{\Delta_\lambda}(\phi) - \tau_{\Delta_{-\lambda}}(\phi)$.

Note that these differences are constant, because in both polygons the functions $\tau_\Delta(\phi)$ and $\alpha_\Delta(\phi)$ are linear according to (2.20). The translation can again be ignored. The rotation tells us that the transition function between the Minkowski coordinates is

$$\boldsymbol{g}_\eta = \mathrm{e}^{-4\pi G M_\eta}, \qquad \boldsymbol{g}_{-\eta} = \mathrm{e}^{4\pi G M_\eta}, \qquad \eta \in \Gamma_\infty, \tag{2.21}$$

where the *energy* M_η is given by

$$8\pi G M_\eta = \alpha_{\Delta_{-\eta}}(\phi) - \alpha_{\Delta_\eta}(\phi). \tag{2.22}$$

This can be evaluated at any point on the links $\pm\eta$, again because of (2.20). We can also take the limit (2.17), where the definition of M_η becomes

$$8\pi G M_\eta = \theta^+_\eta - \theta^-_\eta. \tag{2.23}$$

The abbreviations $\theta^\pm_\eta$ are defined as

$$\theta^+_\eta = -\alpha_{\Delta_\eta}(\phi_\eta), \qquad \theta^-_\eta = -\alpha_{\Delta_{-\eta}}(\phi_\eta). \tag{2.24}$$

Of course, in the limit $G \to 0$, the relation (2.21) reduces to the free particle relation (1.17), stating that the external relative position vector $\boldsymbol{q}_\lambda$ is proportional to $\boldsymbol{\gamma}_0$. To see that the variable M_η can still be interpreted as an energy, consider the definition (2.23). It is the usual relation between the energy of a particle and the deficit angle, if it is measured in a frame which is not the rest frame of the particle [13]. And in fact, the difference on the right hand side defines a certain angle, which can be regarded as a deficit angle.

To see this, we have to consider the unit vectors $\boldsymbol{z}_{\pm\eta}$, defining the directions of the external edges in figure 5. First of all, we observe that a local isometry (2.19) maps a spatial half line in the spinning cone onto a spatial half line in Minkowski space. So, the unit vectors $\boldsymbol{z}_{\pm\eta}$ are still orthogonal to the $\boldsymbol{\gamma}_0$-axis. To find the directions of these vectors, we have to evaluate the right hand side of (2.19) in the limit (2.17), on the external edges $\pm\eta$. This gives

$$\boldsymbol{z}_\eta = \boldsymbol{\gamma}(\phi_\eta + \theta^+_\eta), \qquad \boldsymbol{z}_{-\eta} = -\boldsymbol{\gamma}(\phi_\eta + \theta^-_\eta), \qquad \eta \in \Gamma_\infty. \tag{2.25}$$

Obviously, the difference $\theta^+_\eta - \theta^-_\eta$ is the angle between the edges $-\eta$ and η in the embedding Minkowski space, on the left hand side of figure 5. We may call this the deficit angle of the external link η. It goes to zero in the limit $G \to 0$.

The variables $\theta^\pm_\eta$ are called the *deviations*. They tell us how much the directions of the vectors (2.25) in Minkowski space, on the left hand side of figure 5, deviate from the directions of the

external links in the spinning cone, on the right hand side of the figure 5. There is no counterpart of these variables for the free particles, because the spinning cone as a reference frame is then replaced by the embedding Minkowski space itself, and therefore the deviations are all zero. However, we shall now show that here the deviations are also not independent. They are determined by the other external link variables.

We already have the relation (2.23). Another relation follows immediately from the definition (2.24). If we replace η by η' in the second equation, and use that $\Delta_\eta = \Delta_{-\eta'}$ for two successive external links $\eta, \eta' \in \Gamma_\infty$, then it follows from (2.20) that

$$\theta^-_{\eta'} - \theta^+_\eta = -4GM\,(\phi_{\eta'} - \phi_\eta). \tag{2.26}$$

This relation has a simple geometric interpretation in figure 5. For the polygon Δ to fit into the spinning cone between the spatial half lines with conical coordinates ϕ_η and $\phi_{\eta'}$, the *opening angle*, hence the angle between the external edges η and $-\eta'$ in Minkowski space, must be equal to $1-4GM$ times the opening angle in conical coordinates, thus the difference $\phi_\eta - \phi_\eta$. This is the factor that shows up in (2.15) in front of the angular coordinate, thus it relates a conical angle to a metric angle.

Now, suppose we are given the energies M_η and the directions ϕ_η of the external links in the spinning cone. Then we can solve the system of linear equations (2.23) and (2.26) for the deviations. We have $2\ell_\infty$ equations and $2\ell_\infty$ unknown variables. However, the equations are not all independent. If we add them all, then the deviations drop out. The differences $\phi_{\eta'} - \phi_\eta$ add up to 2π, and what we get is

$$M = \sum_{\eta\in\Gamma_\infty} M_\eta. \tag{2.27}$$

Hence the total energy is again the sum of the energies of the external links. Or, in a geometric language, the deficit angle of the spinning cone on the right hand side of figure 5 is the sum of the deficit angles of the external links on the left hand side.

So, given the energies M_η and the conical directions ϕ_η, the deviations are determined only up to an overall constant $\theta^\pm_\eta \mapsto \theta^\pm_\eta + \theta$, where $\theta \in \mathbb{R}$ is an unknown angle. In figure 5, this means that we are still free to perform an overall rotation of all non-compact polygons in the embedding Minkowski space on the left hand side. We need one extra condition to fix this freedom, so that the deviations are given uniquely as functions of the other link variables. And of course, we want that in the limit $G \to 0$ we get $\theta^\pm_\eta \to 0$.

The extra condition that we impose is the following *average condition*. We want that, in a certain sense, the overall orientation of the Minkowski frame on the left hand side in figure 5 is the same as the overall orientation of the spinning cone on the right hand side. This condition can be formulated as follows. Consider a certain angular direction ϕ in the spinning cone, not necessary the direction of one of the external links. A radial line pointing into this direction is mapped onto a radial line in Minkowski space, which points into the direction $\phi - \alpha_\Delta(\phi)$, where Δ is the polygon that contains this radial line. This can be read off from (2.19).

Hence, $\alpha_\Delta(\phi)$ tells us how much a particular angular direction in Minkowski space deviates from the corresponding angular direction in the spinning cone. The average condition now requires that on average this deviation is zero. More precisely, the integral $\alpha_\Delta(\phi)$ over all angular directions ϕ vanishes. Hence,

$$\sum_{\Delta\in\Pi_\infty} \int_{\phi_\eta}^{\phi_{\eta'}} \mathrm{d}\phi\,\alpha_\Delta(\phi) = 0. \tag{2.28}$$

Here we took into account that each non-compact polygon $\Delta \in \mathit{\Pi}_\infty$ covers a certain range of the conical coordinates, namely the one between the external edges $\eta, \eta' \in \mathit{\Gamma}_\infty$, where $\Delta = \Delta_\eta = \Delta_{-\eta'}$.

This average condition imposes one extra restriction on the Minkowski coordinates in the non-compact charts. To see that it fixes the values of the deviations, let us simplify it a little bit. Since α_Δ is a linear function, we can evaluate the integral in (2.28). The value of the integral of a linear function is half the sum of the values of function at the end points, multiplied by the length of the integration interval. With the irrelevant factor of one half dropped, and the definitions (2.24) inserted, we get

$$\sum_{\Delta \in \mathit{\Pi}_\infty} (\theta^-_{\eta'} + \theta^+_\eta)\,(\phi_{\eta'} - \phi_\eta) = 0. \tag{2.29}$$

We can then also make use of the relations (2.26) and (2.23), and rearrange the summation slightly, which gives the equivalent condition

$$\sum_{\eta \in \mathit{\Gamma}_\infty} M_\eta\,(\theta^+_\eta + \theta^-_\eta) = 0. \tag{2.30}$$

We see that this provides an additional linear equation, which fixed the absolute values of the deviations. In the limit $G \to 0$, it is easy to verify that the unique solution to the system of linear equations (2.23), (2.26), and (2.30) is $\theta^\pm_\eta = 0$.

So, after this somewhat technical derivation, what is the conclusion? The deviations $\theta^\pm_\eta$, and thus the unit vectors $\boldsymbol{z}_{\pm\eta}$ are uniquely specified by the external link variables ϕ_η and M_η. The geometry of the ADM surface and its embedding into the reference frame is finally specified by almost same independent link variables as previously for the free particles. We have the relative position vectors $\boldsymbol{z}_\lambda$, and the transition functions $\boldsymbol{g}_\lambda$ for $\lambda \in \mathit{\Gamma}_+$, and the energies M_η, the clocks T_η, and the directions ϕ_η for $\lambda \in \mathit{\Gamma}_\infty$. We shall later also introduce an auxiliary phase space variable L_η, representing the angular momentum conjugate to ϕ_η, but for the time being we do not need this.

There is then, finally, another consistency condition which involves the clocks. It ensures that the ADM surface can be embedded into the spinning cone, so that the external links are in fact spatial half lines with the given conical time coordinates. Consider again a non-compact polygon $\Delta \in \mathit{\Pi}_\infty$, and the external links $\eta, \eta' \in \mathit{\Gamma}_\infty$ with $\Delta = \Delta_\eta = \Delta_{-\eta'}$. Both edges are spatial half lines in Minkowski space. The constant $\boldsymbol{\gamma}_0$-coordinates of these edges are given by

$$T_\eta + \tau_\Delta(\phi_\eta), \qquad T_{\eta'} + \tau_\Delta(\phi_{\eta'}), \tag{2.31}$$

respectively. This follows again from (2.19), evaluated in the limit (2.17). The difference between these coordinates is called the *time offset* of the polygon Δ. It is given by

$$T_{\eta'} - T_\eta + \tau_\Delta(\phi_{\eta'}) - \tau_\Delta(\phi_\eta) = T_{\eta'} - T_\eta + 4GS(\phi_{\eta'} - \phi_\eta). \tag{2.32}$$

Now, the same difference is given by the sum of the $\boldsymbol{\gamma}_0$-components of the relative position vectors $\boldsymbol{z}_\lambda$ defining the internal edges $\lambda \in \mathit{\Gamma}_\Delta \cap \mathit{\Gamma}_0$ of the polygon Δ. Hence, we have the relation

$$T_{\eta'} - T_\eta + 4GS(\phi_{\eta'} - \phi_\eta) + \sum_{\lambda \in \mathit{\Gamma}_\Delta \cap \mathit{\Gamma}_0} z^0_\lambda = 0. \tag{2.33}$$

This is obviously a generalization of (1.25), to which it reduces in the limit $G \to 0$. We can use it to compute the parameter S of the spinning cone, thus the total angular momentum of the universe. We have to sum over all non-compact polygons in (2.33). Then the differences between the clocks $T_{\eta'} - T_\eta$ drop out, and the conical angles $\phi_{\eta'} - \phi_\eta$ add up to 2π. The result is

$$8\pi G S = - \sum_{\Delta \in \Pi_\infty} \sum_{\lambda \in \Gamma_\Delta \cap \Gamma_0} z_\lambda^0. \tag{2.34}$$

On the right hand side we have to sum over all internal edges of all non-compact polygons. To simplify this, we may equally well sum over all compact polygons as well. For each compact polygon, the sum over its edges is zero, according to (2.18). But then the sum just goes over all internal links, thus

$$S = -\frac{1}{8\pi G} \sum_{\lambda \in \Gamma_0} z_\lambda^0 = \sum_{\lambda \in \Gamma_+} L_\lambda, \quad \text{where} \quad L_\lambda = -\frac{1}{8\pi G}\,(z_\lambda^0 + z_{-\lambda}^0). \tag{2.35}$$

This looks very similar to (1.30). The total angular momentum is distributed over the internal links. To recover the free particle expression for $L_\lambda = L_{-\lambda}$, we use the relation (2.2), which tells us that

$$L_\lambda = \frac{1}{16\pi G}\,\mathrm{Tr}((\boldsymbol{g}_\lambda\, \boldsymbol{z}_\lambda\, \boldsymbol{g}_\lambda^{-1} - \boldsymbol{z}_\lambda)\,\boldsymbol{\gamma}^0). \tag{2.36}$$

We can then write the transition function $\boldsymbol{g}_\lambda$ as an exponential of the relative momentum vector $\boldsymbol{q}_\lambda$, and expand this up to the first order in G as in (2.3). In the leading order in G, we then recover (1.30).

So, we finally see that not only the geometry of the ADM surface and its embedding into the reference frame is determined by the link variables, but also the parameters M and S of the spinning cone. And we have the same kind of consistency conditions, a vector equation (2.18) for every compact polygon, and a scalar equation (2.33) for every non-compact polygon.

Redundancies

In the beginning we already saw that the redundancy transformations of the free particles system are replaced by the coordinate transformations, hence the Lorentz rotations in the Minkowski charts associated with the polygons. We found that the link variables transform as

$$\boldsymbol{g}_\lambda \mapsto \boldsymbol{h}_{\Delta_{-\lambda}}\, \boldsymbol{g}_\lambda\, \boldsymbol{h}_{\Delta_\lambda}^{-1}, \qquad \boldsymbol{z}_\lambda \mapsto \boldsymbol{h}_{\Delta_\lambda}\, \boldsymbol{z}_\lambda\, \boldsymbol{h}_{\Delta_\lambda}^{-1}, \tag{2.37}$$

where $\boldsymbol{h}_\Delta \in \mathsf{SL}(2)$ represents the Lorentz rotation acting on the polygon Δ. There is now a restriction on these parameters, because the Minkowski coordinates in the non-compact charts are related in a certain way to the conical coordinates. We are only allowed to perform spatial rotations, thus for non-compact polygons $\Delta \in \Pi_\infty$ we must have

$$\boldsymbol{h}_\Delta = \mathrm{e}^{-4\pi G \omega_\Delta \boldsymbol{\gamma}_0}, \qquad \Delta \in \Pi_\infty, \tag{2.38}$$

where $\omega_\Delta \in \mathbb{R}$ specifies the angle rotation for the polygon Δ. This is also the restriction that we had for the free particles. The definition (2.38) is chosen so that the energies transform in the same way as before, thus

$$M_\eta \mapsto M_\eta + \omega_{\Delta_{-\eta}} - \omega_{\Delta_\eta}. \tag{2.39}$$

Clearly, the link variables T_η and ϕ_η are invariant under such a transformation, because we only change the local Minkowski coordinates, but not the embedding of the ADM surface into the spinning cone. There is then, however, a transformation of the deviations. For the transformation (2.37) to apply also to the unit vector $\boldsymbol{z}_{\pm\eta}$ in (2.25), we must have

$$\theta^+_\eta \mapsto \theta^+_\eta - 8\pi G\,\omega_{\Delta_\eta}, \qquad \theta^-_\eta \mapsto \theta^-_\eta - 8\pi G\,\omega_{\Delta_{-\eta}}. \tag{2.40}$$

Now, one can easily check that this is consistent with (2.23) and (2.26), hence with the definition of the deviations as functions of the other link variables. But it is not consistent with (2.30). So, there is something wrong with this transformation.

Before we look at this more closely, let us consider a second class of redundancies. As for the free particles, we are free to rotate the external links within a certain range, which includes a smooth deformation of the ADM surface. However, due to the screw like geometry of the spinning cone, a rotation of the external link η not only affects the conical angular coordinate ϕ_η, but also the conical time coordinate T_η. It is not difficult to see that a spatial half line in the spinning cone is always a line of constant $T + 4GS\phi$. Thus if we fix the start point of a spatial half line and rotate it about this point, then its coordinates, in the limit $R \to \infty$, transform as

$$\phi_\eta \mapsto \phi_\eta + \epsilon_\eta, \qquad T_\eta \mapsto T_\eta - 4GS\,\epsilon_\eta, \qquad \eta \in \mathit{\Gamma}_\infty, \tag{2.41}$$

where $\epsilon_\eta \in \mathbb{R}$ defines the angle of rotation in conical coordinates. To see how such a transformation acts on the embedded polygons in Minkowski space, we use the definition (2.24) of the deviations, and the property (2.20) of the functions α_Δ. Note that these functions are now fixed, because we only rotate the external link, but we do not change the transition function between the conical coordinates and the Minkowski coordinates. It follows that

$$\theta^+_\eta \mapsto \theta^+_\eta - 4GM\,\epsilon_\eta, \qquad \theta^-_\eta \mapsto \theta^-_\eta - 4GM\,\epsilon_\eta. \tag{2.42}$$

Together with (2.41), this implies that the unit vectors $\boldsymbol{z}_{\pm\eta}$ are rotated by $(1-4GM)\epsilon_\eta$ about the $\boldsymbol{\gamma}_0$-axis in the embedding Minkowski space. Again, we recover the factor $1-4GM$, which relates a conical angle to the corresponding angle in Minkowski space. And we also see that there is a certain restriction on the parameters ϵ_η, because the non-compact polygons are twisted when the external edges are rotated, and for too large parameters they are no longer spacelike.

One can again verify that the transformations (2.42) are compatible with (2.23) and (2.26), but not with (2.30). To understand this, let consider the infinitesimal generator of the most general redundancy transformation. To define the generator of a Lorentz rotation acting on a compact polygon $\Delta \in \mathit{\Pi}_0$, we introduce a vector $\boldsymbol{\chi}_\Delta \in \mathfrak{sl}(2)$. It is related to the actual parameter of the Lorentz rotation by $\boldsymbol{h}_\Delta = \mathrm{e}^{4\pi G \boldsymbol{\chi}_\Delta}$. For a non-compact polygon $\Delta \in \mathit{\Pi}_\infty$, we set $\boldsymbol{\chi}_\Delta = -\omega_\Delta \boldsymbol{\gamma}_0$ for some $\omega_\Delta \in \mathbb{R}$, which is then also compatible with (2.38). And finally, the generator of a rotation of the external links is defined by ϵ_η, which is also the parameter in (2.41).

For the transition functions and the relative position vectors we find the following variations,

$$\delta \boldsymbol{g}_\lambda = 4\pi G\,(\boldsymbol{\chi}_{\Delta_{-\lambda}}\,\boldsymbol{g}_\lambda - \boldsymbol{g}_\lambda\,\boldsymbol{\chi}_{\Delta_\lambda}), \qquad \delta \boldsymbol{z}_\lambda = 4\pi G\,[\boldsymbol{\chi}_{\Delta_\lambda}, \boldsymbol{z}_\lambda], \qquad \lambda \in \mathit{\Gamma}_0. \tag{2.43}$$

This is the infinitesimal generator of (2.37). For the external link variables we find

$$\delta M_\eta = \omega_{\Delta_{-\eta}} - \omega_{\Delta_\eta}, \qquad \delta T_\eta = -4GS\,\epsilon_\eta, \qquad \delta\phi_\eta = \epsilon_\eta, \qquad \eta \in \mathit{\Gamma}_\infty, \tag{2.44}$$

which is the generator of (2.39) and (2.41), respectively. And finally, for the deviations this implies

$$\delta\theta^{+}_{\eta} = -4GM\,\epsilon_{\eta} - 8\pi G\,\omega_{\Delta_{\eta}}, \qquad \delta\theta^{-}_{\eta} = -4GM\,\epsilon_{\eta} - 8\pi G\,\omega_{\Delta_{-\eta}}. \tag{2.45}$$

In the limit $G \to 0$, these are precisely the transformations (1.43). However, there is one crucial difference. For the free particles, we found that a certain combination of redundancy transformation was trivial. Setting $\epsilon_{\eta} = 0$ and $\boldsymbol{\chi}_{\Delta} = -\omega\boldsymbol{\gamma}_0$ for all polygons, with $\omega \in \mathbb{R}$, we found that the transformation was void. But here we still get a non-trivial transformation, namely

$$\delta\boldsymbol{g}_{\lambda} = 4\pi G\,\omega\,[\boldsymbol{z}_{\lambda}, \boldsymbol{\gamma}_0], \qquad \delta\boldsymbol{z}_{\lambda} = 4\pi G\,\omega\,[\boldsymbol{z}_{\lambda}, \boldsymbol{\gamma}_0], \qquad \delta\theta^{\pm}_{\eta} = -8\pi G\,\omega. \tag{2.46}$$

On the left hand side in figure 5, this is a *simultaneous* rotation of all polygons in the embedding Minkowski space, by $8\pi G\omega$ in clockwise direction about the $\boldsymbol{\gamma}_0$-axis. It is this transformation that is not compatible with the average condition (2.30).

It is now obvious what this means. Since we wanted the overall orientation of the Minkowski frame to be the same as that of the conical frame, we are not allowed to perform such an overall rotation. There is a restriction on the parameters ω_{Δ} and ϵ_{η}. The condition is that the average condition (2.30) must be preserved. Inserting the variation of the energies M_{η} and the deviations $\theta^{\pm}_{\eta}$ from above, one can easily derive the following restriction to be imposed on the parameters ω_{Δ} and ϵ_{η},

$$\sum_{\Delta\in\Pi_{\infty}} \omega_{\Delta}\,(\phi_{\eta'} - \phi_{\eta}) + \sum_{\eta\in\Gamma_{\infty}} \epsilon_{\eta}\,M_{\eta} = 0. \tag{2.47}$$

As for the free particles, we have one redundancy less than there are parameters. It is only a technical difference that this is now due to a restriction on the parameters, whereas for the free particles a certain combination of the parameters drops out. We shall look at this again at the phase space level in section 3, and there we shall see that the analogy to the free particle system is indeed very close.

Inserting and removing links

There is another class of redundancy transformations, which we more or less ignored so far. Of course, we can also change the triangulation of the ADM surface, and this should not affect the state of the particles. If we consider the time evolution, for example, then we have to do this from time to time, since not every configuration of the particles admits same triangulation. For example, if we have a graph Γ where the particles π_{λ} and $\pi_{-\lambda}$ are connected by an internal link λ, then it is in general not possible to use this triangulation for a state where these particles are far apart, with many particles in between.

What typically happens is that a spacelike geodesic connecting these particles intersects with the light cones emerging from the other particles in between. This is actually not a feature introduced by the gravitational interaction. The problem also arises for the free particle system, where we just ignored it. So, let us show how to transform from one triangulation to another. We shall later consider this as a *large* gauge transformation, in contrast to the redundancy transformations above, which are *smoothly generated* gauge symmetries. A convenient way to describe a general transition between two different triangulations is to decompose it into elementary steps.

An elementary step is the insertion or the removal of a single link. Inserting a new internal link is very simple. Suppose we have a graph Γ with an associated collection Π of polygons,

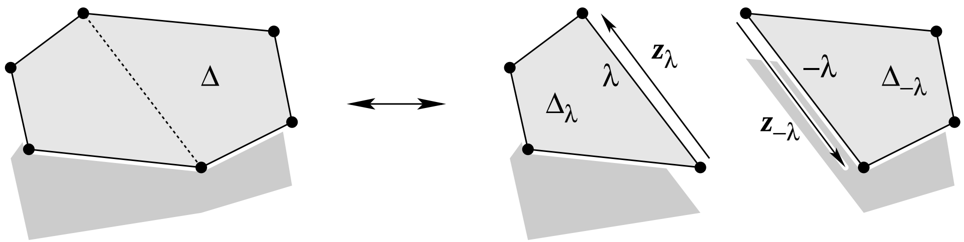

Figure 6: A compact polygon can be split into two compact polygons by inserting a pair of links $\pm\lambda$. The new link variables $\boldsymbol{g}_{\pm\lambda}$ and $\boldsymbol{z}_{\pm\lambda}$ are determined by (2.48) and (2.49). For the reverse transformation, one first has to perform a redundancy transformation, so that $\boldsymbol{g}_{\pm\lambda} = \mathbf{1}$. The two Minkowski charts can then be unified, the polygons can be glued together, and the links $\pm\lambda$ can be removed.

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and let $\Delta \in \Pi$ by a polygon with more than three edges. It can be either a compact or a non-compact polygon. For a compact one, the typical situation in sketched in figure 6. Consider one of the diagonals of this polygon, and suppose that the polygon surface can be deformed, so that the diagonal is contained in the surface. Then it is possible to cut the polygon into two new polygons.

The insertion is defined as follows. The new graph $\tilde{\Gamma} = \Gamma \cup \{\lambda, -\lambda\}$ is obtained by adding a pair of internal links to the original graph Γ. The new collection $\tilde{\Pi}$ of polygons is obtained by removing Δ and adding Δ_λ and $\Delta_{-\lambda}$. The edges $\tilde{\Gamma}_{\Delta_\lambda}$ and $\tilde{\Gamma}_{\Delta_\lambda}$ of the new polygons are defined in the obvious way. The links λ is also added to the set $\tilde{\Gamma}_{\pi_\lambda}$, and $-\lambda$ is inserted into $\tilde{\Gamma}_{\pi_{-\lambda}}$ at the appropriate place in the cyclic ordering. All other subsets of links $\tilde{\Gamma}_\Delta = \Gamma_\Delta$ and $\tilde{\Gamma}_\pi = \Gamma_\pi$ are unchanged.

Regarding the new link variables $\boldsymbol{g}_{\pm\lambda}$ and $\boldsymbol{z}_{\pm\lambda}$, we have to define them in such a way that the geometry of the ADM surface is the same as before. Since both polygons are originally embedded into the same Minkowski chart, it is clear that the newly arising transition function is trivial, hence

$$\boldsymbol{g}_\lambda = \boldsymbol{g}_{-\lambda} = \mathbf{1}. \tag{2.48}$$

The relative position vector $\boldsymbol{z}_\lambda$ is determined as follows. At least one of the newly arising polygons is compact, even if the original polygon Δ was non-compact. Without loss of generality, let us assume that Δ_λ is compact. Hence, $\tilde{\Gamma}_{\Delta_\lambda}$ contains the edge λ, and a sequence of at least two other internal edges. The new vectors $\boldsymbol{z}_{\pm\lambda}$ are then given by

$$\boldsymbol{z}_\lambda = -\boldsymbol{z}_{-\lambda} = -\sum_{\kappa \in \tilde{\Gamma}_{\Delta_\lambda} - \{\lambda\}} \boldsymbol{z}_\kappa. \tag{2.49}$$

This immediately implies that the consistency condition (2.18) for the new polygon Δ_λ is satisfied. Moreover, if the original polygon Δ was compact, the the same holds for the new compact polygon $\Delta_{-\lambda}$. This follows from (2.49) and the consistency condition (2.18) for the original polygon. Similarly, if the original polygon Δ was non-compact, the the new polygon $\Delta_{-\lambda}$ is also

non-compact, and the consistency condition (2.33) for the new polygon follows from the same consistency condition for the original polygon.

So, the conclusion is that, whenever it is possible to deform the ADM surface so that it contains the diagonal of a polygon, then it is possible to add this as a new pair of internal links to the graph. If we define the new link variables according to (2.48) and (2.49), then the new triangulation defines the same local geometry of the ADM surface, up to smooth deformations, and thus the same physical state. For the free particle system, the insertion of a new internal link works in the very same way, we just have to replace (2.48) by $\boldsymbol{q}_\lambda = \boldsymbol{q}_{-\lambda} = 0$.

Inserting a new external link into a non-compact polygon is equally straightforward. Suppose that we have a graph $\varGamma$ and $\varGamma_\infty = \{\ldots, \eta, \eta', \ldots\}$ is the set of external links. Suppose further that the non-compact polygon $\Delta_\eta = \Delta_{-\eta'}$ has more than one internal edge. Provided that the polygon surface can be deformed appropriately, we can then introduce a new external link ϱ between η and η', so that $\tilde{\varGamma}_\infty = \{\ldots, \eta, \varrho, \eta', \ldots\}$. The original polygon $\Delta_\eta = \Delta_{-\eta'}$ splits into two new non-compact polygons $\Delta_\eta = \Delta_{-\varrho}$ and $\Delta_\varrho = \Delta_{-\eta'}$.

The new link variables associated with ϱ are determined as follows. In contrast to the insertion of a new internal link, there is now some freedom regarding the direction of the new link. The new conical direction ϕ_ϱ can be any direction in the range

$$\phi_\eta \le \phi_\varrho \le \phi_{\eta'}. \tag{2.50}$$

It has to lie between the original links η and η', but we are free to rotate it, because there is no fixed end point. But once the conical direction ϕ_ϱ is chosen, all other link variables are fixed. For the same reason as before, we have to set

$$M_\varrho = 0 \quad \Rightarrow \quad \boldsymbol{g}_\varrho = \boldsymbol{g}_{-\varrho} = \mathbf{1}, \tag{2.51}$$

The transition function between the two new Minkowski charts is trivial. The clock T_ϱ is determined by the consistency condition (2.33), which has to be satisfied for both new polygons Δ_ϱ and $\Delta_{-\varrho}$, thus

$$T_\varrho = T_\eta - 4GS(\phi_\varrho - \phi_\eta) - \sum_{\lambda \in \tilde{\varGamma}_{\Delta_{-\varrho}} \cap \tilde{\varGamma}_0} z^0_\lambda = T_{\eta'} + 4GS(\phi_{\eta'} - \phi_\varrho) + \sum_{\lambda \in \tilde{\varGamma}_{\Delta_\varrho} \cap \tilde{\varGamma}_0} z^0_\lambda. \tag{2.52}$$

The last equality is the consistency condition (2.33) for the original polygon $\Delta_\eta = \Delta_{-\eta'}$, so we may use either expression to define T_ϱ.

Finally, consider the deviations $\theta^\pm_\varrho$. Since they are determined by system of linear equations, involving all external links, it is not immediately obvious whether inserting a new external link affects the deviations of the other external links or not. So, consider the system of linear equations (2.23), (2.26), and (2.30), after the insertion, and let us write down only those equation that involve the new links $\pm\varrho$. First of all, we see that the relevant term in the average condition (2.30) drops out, because $M_\varrho = 0$. So, at least this equation is unaffected.

There are then two equations of the form (2.26), namely those that define the opening angles of the polygons $\Delta_{\pm\varrho}$,

$$\theta^-_{\eta'} - \theta^+_\varrho = -4GM\,(\phi_{\eta'} - \phi_\varrho), \qquad \theta^-_\varrho - \theta^+_\eta = -4GM\,(\phi_\varrho - \phi_\eta), \tag{2.53}$$

and one equation of the form (2.23), namely that for the deficit angle of the new link,

$$\theta^{+}_{\varrho} - \theta^{-}_{\varrho} = 8\pi G M_{\varrho} = 0. \tag{2.54}$$

First of all, we see that once we chose the conical direction ϕ_{ϱ} of the new link, these equations can be used to determine the deviations $\theta^{\pm}_{\eta}$. Moreover, if we then eliminate the variables $\theta^{\pm}_{\eta}$ from three given equations above, we get

$$\theta^{-}_{\eta'} - \theta^{+}_{\eta} = -4GM\,(\phi_{\eta'} - \phi_{\eta}). \tag{2.55}$$

But this is exactly the equation (2.26) for the original polygon $\Delta_{\eta} = \Delta_{-\eta}$, before the insertion. Hence, it follows that the other external links are not affected by the insertion.

So, we can insert internal and external links, whenever it is possible to deform the polygons appropriately so that the new link becomes a spacetime geodesic contained in the ADM surface. And what about removing links? Suppose that $\pm\lambda$ is a pair of internal or external links, so that $\Delta_{\lambda} \neq \Delta_{-\lambda}$. Hence the links do not represent a self overlap region of a single coordinate chart. Then it is possible to perform a coordinate transformation in one chart, so that the transition function $\boldsymbol{g}_{\lambda} = \boldsymbol{g}_{-\lambda} = \mathbf{1}$ becomes trivial. Note that this is not possible if $\Delta_{\lambda} = \Delta_{-\lambda}$, because then the transformation (2.37) is a conjugation, which cannot be used to transform a group element into the unit element.

So, in order to remove a pair of links, we first have to perform a smoothly generated redundancy transformation, or *fix a gauge*, so that the associated transition function becomes trivial. Then the link can be removed, and the two Minkowski charts can be unified into a single one. We can remove as many links as we like, until there is only a single polygon left, and the graph is a tree. Vice versa, the maximal number of links is reached when all polygons are triangles. And finally, we can transform between any two triangulations using the given elementary steps, and the previously considered smoothly generated redundancy transformations, thus the Lorentz rotations of the polygons and rotations of the external links.

3 Phase space geometry

After this somewhat technical derivation, let us now come to the actual subject of this article, the phase space structure of the multi particle model. In the previous section we have seen how the geometry of the ADM surface, and its embedding into the reference frame, is specified by a triangulation and a set of link variables. It is possible to set up the same hierarchy of phase spaces, which we also encountered in section 1 for the free particles,

$$\begin{array}{ccccc} \mathcal{Q} & \supset & \mathcal{S} & \supset & \mathcal{P} \\ & & \downarrow & & \downarrow \\ & & \mathcal{S}/\!\sim & & \mathcal{P}/\!\sim. \end{array} \tag{3.1}$$

The *extended* phase space $\mathcal{Q}$ is spanned by the link variables. The *kinematical* subspace $\mathcal{S} \subset \mathcal{Q}$ is defined by a set of kinematical constraints. They are the consistency conditions derived in the previous section, and the associated gauge symmetries are the redundancy transformation. The quotient space $\mathcal{S}/\!\sim$ is basically the set of all possible geometries of the ADM surface, and all possible embeddings into the reference frame.

In analogy to general relativity, we can think of the extended phase space $\mathcal{Q}$ as spanned by the usual ADM variables, the spatial metric and its conjugate momentum, the external curvature. The kinematical constraints are then the diffeomorphism constraints, the associated gauge symmetries are the coordinates transformation on the space manifold, and consequently the quotient space is, roughly speaking, the set of all spatial geometries. Following this analogy, we can then regard the mass shell constraints as a the Hamiltonian constraints of general relativity. They define the *physical* subspace $\mathcal{P} \subset \mathcal{S}$, and the quotient space $\mathcal{P}/\!\sim$ is the set of all spacetimes, or the set of all physically inequivalent solutions to the equations of motion.

As for the free particles, the phase space consists of a finite number of disconnected components, one for each possible graph $\varGamma$ with n particles,

$$\mathcal{Q} = \bigcup_{\varGamma} \mathcal{Q}_{\varGamma}, \qquad \mathcal{S} = \bigcup_{\varGamma} \mathcal{S}_{\varGamma}, \qquad \mathcal{P} = \bigcup_{\varGamma} \mathcal{P}_{\varGamma}. \tag{3.2}$$

To make the quotient spaces $\mathcal{S}/\!\sim$ and $\mathcal{P}/\!\sim$ connected manifold, we have to include large gauge transformations into the definition of the equivalence relation, and these are of course the transitions between the triangulations, as defined in the very end of the previous section. In the context of general relativity they are analogous to the large diffeomorphism.

The whole phase space structure can be regarded as a *deformed* version of the free particle phase space, again with Newton's constant as a deformation parameter. At any stage of the derivation we can get back to the free particles by taking the limit $G \to 0$. Using this we can see how the phase space structures are actually affected when the gravitational interaction is switched on. The most interesting deformation is that of the symplectic structure, which has been mentioned in the introduction. The Poisson brackets of the components of the relative position vectors are no longer zero, which means that when the system is quantized these become non-commuting objects.

The extended phase space

The definition of the extended phase space $\mathcal{Q}$ is almost the same as before. There are finitely many disconnected components $\mathcal{Q}_{\varGamma}$. Each component is specified by a graph $\varGamma$ with n particles, and it is spanned by the following link variables,

- a *transition function* $\boldsymbol{q}_{\lambda} \in \mathsf{SL}(2)$ and a *relative position vector* $\boldsymbol{z}_{\lambda} \in \mathfrak{sl}(2)$ for every internal link $\lambda \in \varGamma_0$, and
- an *energy* M_{η}, a *clock* T_{η}, an *angular momentum* L_{η}, and a *direction* ϕ_{η} for every external link.

The only modification, as compared to the free particles, is that the relative momentum vector $\boldsymbol{q}_{\lambda}$ is replaced by the transition function $\boldsymbol{g}_{\lambda}$. There is still a relation between the variables associated with an internal link λ and the reversed link $-\lambda$. According to (2.2) we have

$$\boldsymbol{g}_{-\lambda} = \boldsymbol{g}_{\lambda}^{-1}, \qquad \boldsymbol{z}_{-\lambda} = -\boldsymbol{g}_{\lambda}\, \boldsymbol{z}_{\lambda}\, \boldsymbol{g}_{\lambda}^{-1}, \qquad \lambda \in \varGamma_0. \tag{3.3}$$

Thus actually only half of the internal link variables are independent. There is also a transition function $\boldsymbol{g}_{\eta}$ for each external link, which is given by

$$\boldsymbol{g}_{\eta} = \mathrm{e}^{-4\pi G M_{\eta} \boldsymbol{\gamma}_0}, \qquad \boldsymbol{g}_{-\eta} = \mathrm{e}^{4\pi G M_{\eta} \boldsymbol{\gamma}_0}, \qquad \eta \in \varGamma_{\infty}. \tag{3.4}$$

And finally we have also defined a vector $\boldsymbol{z}_\eta$ for each external link, which is a spatial unit vector given by

$$\boldsymbol{z}_\eta = \boldsymbol{\gamma}(\phi_\eta + \theta^+_\eta), \qquad \boldsymbol{z}_{-\eta} = -\boldsymbol{\gamma}(\phi_\eta + \theta^-_\eta), \qquad \eta \in \Gamma_\infty. \tag{3.5}$$

The quantities $\theta^\pm_\eta$, which we called the *deviations*, are implicitly given by the following system of linear equations

$$\theta^-_{\eta'} - \theta^+_\eta = -4GM\,(\phi_{\eta'} - \phi_\eta), \qquad \theta^+_\eta - \theta^-_\eta = 8\pi G M_\eta, \tag{3.6}$$

and the *average condition*

$$\sum_{\eta\in\Gamma_\infty} M_\eta\,(\theta^+_\eta + \theta^-_\eta) = 0. \tag{3.7}$$

This implies that the relation (3.3) also holds for external links. Provided that some consistency conditions are satisfied by the link variables, which we shall impose below as kinematical constraints, it is then possible to construct the polygons in figure 3, glue them together, and embed the resulting ADM surface into the spinning cone, as indicated in figure 5. This spinning cone defines the centre of mass frame, and also the reference frame. The geometry of the spinning cone depends on two parameters M and S. They represent the total energy and the total angular momentum of the universe, and they are given by

$$M = \sum_{\eta\in\Gamma_\infty} M_\eta, \qquad S = \sum_{\lambda\in\Gamma_+} L_\lambda, \quad \text{where} \quad L_\lambda = -\frac{1}{8\pi G}\,(z^0_\lambda + z^0_{-\lambda}). \tag{3.8}$$

This is the definition of the extended phase space $\mathcal{Q}_\Gamma$. It is a $6\ell_0 + 4\ell_\infty$ dimensional manifold. It is a *deformed* version of the free particle phase space in the sense that the vectors $\boldsymbol{q}_\lambda \in \mathfrak{sl}(2)$ are replaced by the group elements $\boldsymbol{g}_\lambda \in \mathsf{SL}(2)$. Using the relation (2.3) in the limit $G \to 0$, we see how this deformation disappears when the gravitational interaction is switched off.

Non-commutative coordinates

Now need to know what the symplectic structure is, and to say something about the dynamics of the particles we also need to know the Hamiltonian. Both will be derived in section 4 from the Einstein Hilbert action. So, let us here only give a definition without any further motivation. The symplectic structure is in fact the most natural one that can be defined on the given phase space. It also agrees very nicely with the symplectic structure derived for the single particle model in [13], of which this multi particle model is a straightforward generalization.

The external link variables still provide two canonical pairs (M_η, T_η) and (L_η, ϕ_η) for of each external link $\eta \in \Gamma_\infty$. For each internal link $\lambda \in \Gamma_0$, the pair $(\boldsymbol{g}_\lambda, \boldsymbol{z}_\lambda)$ can be regarded as an element of the cotangent bundle of the group manifold $\mathsf{T}_*\mathsf{SL}(2) = \mathsf{SL}(2) \times \mathfrak{sl}(2)$. The most natural symplectic structure is therefore the canonical symplectic structure on this cotangent bundle, which is fixed up to a constant. This constant has to be proportional to the inverse of Newton's constant, in order to provide the correct physical dimension of the symplectic potential, which is that of an action. The numerical factor can then be derived from the condition that in the limit $G \to 0$ we have to recover the free particle expression (1.35).

All together, this implies that the most natural symplectic potential on the extended phase space $\mathcal{Q}_\varGamma$ is

$$\varTheta = \sum_{\eta\in\varGamma_\infty} (T_\eta \,\mathrm{d}M_\eta + L_\eta \,\mathrm{d}\phi_\eta) - \frac{1}{8\pi G} \sum_{\lambda\in\varGamma_+} \mathrm{Tr}(\boldsymbol{g}_\lambda^{-1} \,\mathrm{d}\boldsymbol{g}_\lambda \,\boldsymbol{z}_\lambda). \tag{3.9}$$

To see that it reduces to (1.35) in the limit $G \to 0$, we have write the transition function $\boldsymbol{g}_\lambda$ as an exponential (2.3) of the relative momentum vector $\boldsymbol{q}_\lambda$, and expand the result up to the first order in G. The given expression is a well defined one-form on $\mathcal{Q}_\varGamma$, because it is independent of the chosen decomposition $\varGamma_0 = \varGamma_+ \cup \varGamma_-$. Using the relations (3.3), we easily see that

$$\mathrm{Tr}(\boldsymbol{g}_{-\lambda}^{-1} \,\mathrm{d}\boldsymbol{g}_{-\lambda} \,\boldsymbol{z}_{-\lambda}) = \mathrm{Tr}(\boldsymbol{g}_\lambda^{-1} \,\mathrm{d}\boldsymbol{g}_\lambda \,\boldsymbol{z}_\lambda). \tag{3.10}$$

So, we find that the free particle symplectic potential has a natural generalization. But we should emphasize that this is just an *ad hoc* definition. The only way to derive this expression is from the Einstein Hilbert action, and this is what we are going to do in section 4.

Let us then derive the Poisson brackets. These are of course the most interesting objects when we want to learn something about the quantized model, without actually performing the quantization. Nothing particular happens for the external link variables,

$$\{T_\eta, M_\eta\} = 1, \qquad \{L_\eta, \phi_\eta\} = 1, \qquad \eta \in \varGamma_\infty. \tag{3.11}$$

More interesting are the internal links. Here we can directly generalize the results from the single particle system. For each internal link, we have a pair $(\boldsymbol{g}_\lambda, \boldsymbol{z}_\lambda)$, with the symplectic potential being the same as (4.3) in [13]. We just have to generalize the Poisson brackets accordingly, which are given by (4.11) in [13], and insert Newton's constant. The vector $\boldsymbol{z}_\lambda$ generates on the group element $\boldsymbol{g}_\lambda$ a multiplication from the right, and $\boldsymbol{z}_{-\lambda}$ generates a multiplication from the left,

$$\{\boldsymbol{g}_\lambda, z_\lambda^a\} = 4\pi G \,\boldsymbol{g}_\lambda \boldsymbol{\gamma}^a, \qquad \{\boldsymbol{g}_\lambda, z_{-\lambda}^a\} = -4\pi G \,\boldsymbol{\gamma}^a \boldsymbol{g}_\lambda, \qquad \lambda \in \varGamma_0. \tag{3.12}$$

To avoid confusion, we shall here and in the following use the vector component notation inside the Poisson bracket, and never write brackets with more than one matrix entry. Since the brackets between different components of the group elements $\boldsymbol{g}_\lambda$ are all zero, it is sufficient to expand the vectors $\boldsymbol{z}_\lambda = z_\lambda^a \,\boldsymbol{\gamma}_a$. The vector components themselves provide a representation of the Lorentz algebra,

$$\{z_\lambda^a, z_\lambda^b\} = 8\pi G \,\varepsilon^{ab}{}_c \,z_\lambda^c, \qquad \lambda \in \varGamma_0. \tag{3.13}$$

The brackets between the components of $\boldsymbol{z}_\lambda$ and $\boldsymbol{z}_{-\lambda}$ are zero, because the generators of left and right multiplication in (3.12) commute. The brackets (3.11–3.13) are therefore the only non-vanishing brackets involving the basic phase space variables. As a cross check, one can verify that the brackets are compatible with (3.3). We can either consider $\boldsymbol{g}_\lambda$ and $\boldsymbol{z}_\lambda$ as a pair of independent phase space variables, or $\boldsymbol{g}_{-\lambda}$ and $\boldsymbol{z}_{-\lambda}$.

The crucial difference to the free particle system is obviously that the components of the relative position vectors $\boldsymbol{z}_\lambda$ have non-vanishing Poisson brackets with each other. At the classical level, this is just a special feature of the symplectic structure. But at the quantum level it implies that the components of the relative position vectors of the particles do not commute. The particles are effectively moving in a kind of *non-commutative* spacetime. We can also say that the geometry of the space manifold defined by the polygons in figure 3 becomes a *non-commutative* geometry.

What this means to a quantized point particle, and the quantum spacetime that this particle effectively sees, has been studied in some detail for the single particle model in [13]. Since we are here not going to quantize the model, we can only say that some qualitatively similar effects are expected for a multi particle system. The technical details are however more involved, because for a proper quantization we also have to solve the various constraints defined below. For a two particle system, this can still be done explicitly. And in fact, one finds some interesting features of the quantized model. For example, it is impossible to localize the particles at a point in space, and it is also impossible to bring them closer together than a certain minimal distance, which is of the order of the Planck length [21].

Kinematical constraints

The kinematical subspace $\mathcal{S}_\Gamma \subset \mathcal{Q}_\Gamma$ is defined in the same way as for the free particles. We have to impose the consistency conditions derived in the previous section as kinematical constraints. For each compact polygon $\Delta \in \Pi_0$, we found the relation (2.18), stating that the edges form a piecewise straight, closed curve in Minkowski space,

$$\boldsymbol{Z}_\Delta = \sum_{\lambda \in \Gamma_\Delta} \boldsymbol{z}_\lambda \approx 0, \qquad \Delta \in \Pi_0. \tag{3.14}$$

For each non-compact polygon $\Delta \in \Pi_\infty$, there was a consistency condition (2.33). It relates the $\boldsymbol{\gamma}_0$-components of the vectors representing the internal edges $\lambda \in \Gamma_\Delta \cap \Gamma_0$ to the conical coordinates of the two external edges $\eta \in \Gamma_\Delta \cap \Gamma_\infty$ and $-\eta' \in \Gamma_\Delta \cap \Gamma_{-\infty}$,

$$\mathcal{Z}_\Delta = T_{\eta'} - T_\eta + 4GS(\phi_{\eta'} - \phi_\eta) + \sum_{\lambda \in \Gamma_\Delta \cap \Gamma_0} z^0_\lambda \approx 0, \qquad \Delta \in \Pi_\infty. \tag{3.15}$$

Finally, there is also a constraint which defines the value of the angular momentum L_η, which is introduced as an auxiliary variable to obtain a well defined phase space. The constraint is a deformed version of (1.40),

$$\mathcal{J}_\eta = L_\eta + 4GSM_\eta \approx 0, \qquad \eta \in \Gamma_\infty. \tag{3.16}$$

At this point, there is no particular motivation for this relation between the angular momentum L_η and the energies M_η. This is what comes out in section 4, from the phase space reduction applied to the Einstein Hilbert action. The only possible motivation for this particular constraint is that the associated gauge symmetry is a rotation (2.41) of the external link η in the spinning cone, and that in the limit $G \to 0$ it reduces to the free particle constraint (1.40).

This applies to all the kinematical constraints. There is also a relation between them, which is the same as (1.44). If we add the $\boldsymbol{\gamma}_0$-components of the vector constraints $\boldsymbol{Z}_\Delta$ for all $\Delta \in \Pi_0$, and the scalar constraints $\mathcal{Z}_\Delta$ for all $\Delta \in \Pi_\infty$, then we get

$$\sum_{\Delta \in \Pi_0} Z^0_\Delta + \sum_{\Delta \in \Pi_\infty} \mathcal{Z}_\Delta = 8\pi GS + \sum_{\lambda \in \Gamma_0} z^0_\lambda = 0. \tag{3.17}$$

The first equality holds because the conical time differences $T_{\eta'} - T_\eta$ add up to zero, and the conical angles $\phi_{\eta'} - \phi_\eta$ add up to 2π. What remains is a sum of the $\boldsymbol{\gamma}_0$-components of the vectors

$\boldsymbol{z}_\lambda$ for all internal links. But this is just the definition of the angular momentum S in (3.8). So, there is one independent kinematical constraint less then the number of equations given by (3.14-3.16). We'll do a precise counting later on.

All kinematical constraint are first class constraints. This is now less trivial because they involve non-commuting phase space variables. But a straightforward calculation shows that they form a closed algebra. Every bracket between two constraints is again a linear combination of constraints. The only interesting bracket is

$$\{Z_\Delta^a, Z_\Delta^b\} = 8\pi G\, \varepsilon^{ab}{}_c\, Z_\Delta^c, \qquad \Delta \in \Pi_0. \tag{3.18}$$

For each compact polygon Δ, the components of the vector constraint $\boldsymbol{Z}_\Delta$ provide a representation of the Lorentz algebra. This is quite reasonable, since we expect the associated gauge symmetries to be the Lorentz rotations of the polygons in the embedding Minkowski space, thus the coordinate transformations in the Minkowski charts.

To derive the associated gauge symmetries, it is again useful to define a general linear combination of the kinematical constraints. We introduce a vector valued multiplier $\boldsymbol{\chi}_\Delta \in \mathfrak{sl}(2)$ for each compact polygon $\Delta \in \Pi_0$, a scalar multiplier $\omega_\Delta \in \mathbb{R}$ for each non-compact polygon $\Delta \in \Pi_\infty$, and another scalar multiplier $\epsilon_\eta \in \mathbb{R}$ for each external link $\eta \in \Gamma_\infty$. Then we define a linear combination like (1.41),

$$\mathcal{K} = \tfrac{1}{2} \sum_{\Delta\in\Pi_0} \mathrm{Tr}(\boldsymbol{\chi}_\Delta \boldsymbol{Z}_\Delta) + \sum_{\Delta\in\Pi_\infty} \omega_\Delta\, \mathcal{Z}_\Delta + \sum_{\eta\in\Gamma_\infty} \epsilon_\eta\, \mathcal{J}_\eta. \tag{3.19}$$

Inserting the constraints, and rearranging the summation, this can be written as

$$\begin{aligned} \mathcal{K} = {} & \tfrac{1}{2} \sum_{\lambda\in\Gamma_0} \mathrm{Tr}(\boldsymbol{\chi}_{\Delta_\lambda} \boldsymbol{z}_\lambda) - \sum_{\eta\in\Gamma_\infty} (\omega_{\Delta_\eta} - \omega_{\Delta_{-\eta}})\, T_\eta + \sum_{\eta\in\Gamma_\infty} \epsilon_\eta\, L_\eta + \\ & + 4GS \Big(\sum_{\Delta\in\Pi_\infty} \omega_\Delta (\phi_{\eta'} - \phi_\eta) + \sum_{\eta\in\Gamma_\infty} \epsilon_\eta\, M_\eta \Big). \end{aligned} \tag{3.20}$$

Again, we defined $\boldsymbol{\chi}_\Delta = -\omega_\Delta \boldsymbol{\gamma}_0$ for non-compact polygons $\Delta \in \Pi_\infty$, because the first sum also involves contributions from the non-compact polygons. The first line is the free particle expression (1.42). The second line is the total contribution that is proportional to S, which comes from both (3.15) and (3.16). It disappears in the limit $G \to 0$.

Since S is also a function of the link variables, the constraints are no longer linear, and this makes things a little bit more complicated. But we can use the following trick. According to the identity (3.17), there is some redundancy in the definition of the multipliers. If we make the replacement $\omega_\Delta \mapsto \omega_\Delta + \omega$ for $\Delta \in \Pi_\infty$, and $\boldsymbol{\chi}_\Delta \mapsto \boldsymbol{\chi}_\Delta - \omega \boldsymbol{\gamma}_0$ for $\Delta \in \Pi_0$, where $\omega \in \mathbb{R}$ is some fixed real number, then the linear combination (3.19) is unchanged. This can be verified explicitly in (3.20), where we once again recover the definition of S when we make this replacement. We can therefore, without loss of generality, choose the multipliers so that

$$\sum_{\Delta\in\Pi_\infty} \omega_\Delta (\phi_{\eta'} - \phi_\eta) + \sum_{\eta\in\Gamma_\infty} \epsilon_\eta\, M_\eta = 0. \tag{3.21}$$

Doing so, the expression for $\mathcal{K}$ simplifies to

$$\mathcal{K} = \tfrac{1}{2} \sum_{\lambda\in\Gamma_0} \mathrm{Tr}(\boldsymbol{\chi}_{\Delta_\lambda} \boldsymbol{z}_\lambda) - \sum_{\eta\in\Gamma_\infty} (\omega_{\Delta_\eta} - \omega_{\Delta_{-\eta}})\, T_\eta + \sum_{\eta\in\Gamma_\infty} \epsilon_\eta\, L_\eta, \tag{3.22}$$

which is just the free particle expression. Note that this is still the most general linear combination of the kinematical constraints, although we are no longer free to choose all multipliers independently.

It is then straightforward to derive the following Poisson brackets of the phase space variables with $\mathcal{K}$. For the internal link variables, we have to apply the brackets (3.12) and (3.13), and what we get is

$$\{\mathcal{K}, \boldsymbol{g}_\lambda\} = 4\pi G\,(\boldsymbol{\chi}_{\Delta_{-\lambda}}\,\boldsymbol{g}_\lambda - \boldsymbol{g}_\lambda\,\boldsymbol{\chi}_{\Delta_\lambda}), \qquad \{\mathcal{K}, \boldsymbol{z}_\lambda\} = 4\pi G\,[\boldsymbol{\chi}_{\Delta_\lambda}, \boldsymbol{z}_\lambda], \qquad \lambda \in \mathit{\Gamma}_0. \tag{3.23}$$

This is obviously the infinitesimal generator (2.43) of a Lorentz rotation of the polygons in the embedding Minkowski space. For the external link variables, we use the brackets (3.11) and find

$$\begin{aligned} \{\mathcal{K}, M_\eta\} &= \omega_{\Delta_{-\eta}} - \omega_{\Delta_\eta}, & \{\mathcal{K}, T_\eta\} &= -4GS\,\epsilon_\eta, \\ \{\mathcal{K}, L_\eta\} &= -4GS(\omega_{\Delta_{-\eta}} - \omega_{\Delta_\eta}), & \{\mathcal{K}, \phi_\eta\} &= \epsilon_\eta, \qquad \eta \in \mathit{\Gamma}_\infty. \end{aligned} \tag{3.24}$$

These are the infinitesimal generators (2.44), with the appropriate transformation of the auxiliary variables L_η added, so that the constraints (3.16) are preserved.

We conclude that the vector constraint $\boldsymbol{Z}_\Delta$ generates a Lorentz rotations of the compact polygon $\Delta \in \mathit{\Pi}_0$, the scalar constraint $\mathcal{Z}_\Delta$ generates a rotation about the $\boldsymbol{\gamma}_0$-axis of the non-compact polygon $\Delta \in \mathit{\Pi}_\infty$, and the auxiliary constraint $\mathcal{J}_\eta$ generates a rotations of the external link $\eta \in \mathit{\Gamma}_\infty$ in the spinning cone. And we also recover the restriction (2.47) on the parameters of these gauge symmetries, which is the same as (3.21). This ensures that the average condition (3.7) is preserved, hence the overall orientation of the Minkowski frame and the conical frame in figure 5.

Mass shell constraints

The definition of the physical phase space $\mathcal{P}_\Gamma$ is also analogous to the free particle system, but technically again a little bit more involved. We have to replace the mass shell constraints by the appropriate deformed versions. First we recall the definition (2.6) of the *holonomy* of the particle π, evaluated in an adjacent polygon Δ,

$$\boldsymbol{u}_{\pi,\Delta} = \prod_{\lambda \in \mathit{\Gamma}_{\pi,\Delta}} \boldsymbol{g}_\lambda, \tag{3.25}$$

and also the definition (2.9) of the momentum vector, which is the projection of the holonomy,

$$\boldsymbol{u}_{\pi,\Delta} = u_\pi\,\mathbf{1} + 4\pi G\,p^a_{\pi,\Delta}\,\boldsymbol{\gamma}_a, \qquad \boldsymbol{p}_{\pi,\Delta} = p^a_{\pi,\Delta}\,\boldsymbol{\gamma}_a. \tag{3.26}$$

They transform under coordinate transformations in the Minkowski charts according to (2.7). This is now expressed in the following brackets with $\mathcal{K}$,

$$\{\mathcal{K}, \boldsymbol{u}_{\pi,\Delta}\} = 4\pi G\,[\boldsymbol{\chi}_\Delta, \boldsymbol{u}_{\pi,\Delta}], \qquad \{\mathcal{K}, \boldsymbol{p}_{\pi,\Delta}\} = 4\pi G\,[\boldsymbol{\chi}_\Delta, \boldsymbol{p}_{\pi,\Delta}]. \tag{3.27}$$

The physical phase space $\mathcal{P}_\Gamma \subset \mathcal{S}_\Gamma$ is defined as a subset of the kinematical phase space, by imposing the mass shell constraints and positive energy conditions (2.10),

$$u_\pi = \cos(4\pi G m_\pi), \qquad p^0_\pi = \tfrac{1}{2}\mathrm{Tr}(\boldsymbol{u}_\pi \boldsymbol{\gamma}^0) > 0. \tag{3.28}$$

As for the free particles, the Hamiltonian becomes a linear combination of the mass shell constraints. To obtain the correct limit $G \to 0$, we have to rescale the constraints by a certain power of G,

$$\mathcal{H} = \sum_{\pi} \zeta_\pi \, \mathcal{C}_\pi, \qquad \mathcal{C}_\pi = \frac{u_\pi - \cos(4\pi G m_\pi)}{16\pi^2 G^2} \approx 0. \tag{3.29}$$

To see that this provides the correct limit $G \to 0$, one has to use the relation (2.13) between the scalar u_π and the momentum vector $\boldsymbol{p}_\pi$, and expand the cosine up to second order in G.

It is also not difficult to see that the mass shell constraints are still first class constraints. The trace of the holonomy u_π commutes with $\mathcal{K}$, and thus with all kinematical constraints. And the scalars u_π also commute with each other, because they only depend on the transition functions $\boldsymbol{g}_\lambda$ and the energies M_η. To derive the time evolution equations, we have to find the brackets of u_π with the other phase space variables. The only variables that do not commute with the mass shell constraints are the relative position vectors $\boldsymbol{z}_\lambda$ for $\lambda \in \mathit{\Gamma}_0$, and the clocks T_η for $\eta \in \mathit{\Gamma}_\infty$.

Consider first the relative position vectors. We have to derive the brackets

$$\{u_\pi, z_\lambda^a\} = \tfrac{1}{2}\{\mathrm{Tr}(\prod_{\kappa \in \mathit{\Gamma}_\pi} \boldsymbol{g}_\kappa), z_\lambda^a\}. \tag{3.30}$$

This bracket is zero unless $\lambda \in \mathit{\Gamma}_\pi$ or $-\lambda \in \mathit{\Gamma}_\pi$. Consider the case $\lambda \in \mathit{\Gamma}_\pi$ first. The transition function $\boldsymbol{g}_\lambda$ is then one of the factors in the product, and according to (3.12) the action of z_λ^a is to insert a factor of $4\pi G \boldsymbol{\gamma}^a$ into the product, behind the factor $\boldsymbol{g}_\lambda$. So, what we have to do is to break up the cyclic product behind the factor $\boldsymbol{g}_\lambda$, and insert a gamma matrix. The result is

$$\{u_\pi, z_\lambda^a\} = 2\pi G \, \mathrm{Tr}(\boldsymbol{\gamma}^a \prod_{\kappa \in \mathit{\Gamma}_{\pi,\Delta_\lambda}} \boldsymbol{g}_\kappa) = 2\pi G \, \mathrm{Tr}(\boldsymbol{\gamma}^a \, \boldsymbol{u}_{\pi,\Delta_\lambda}) = 16\pi^2 G^2 \, p^a_{\pi,\Delta_\lambda}. \tag{3.31}$$

If we have $-\lambda \in \mathit{\Gamma}_\pi$ instead, then we have to use the second bracket in (3.12). It is then a factor of $-4\pi G \boldsymbol{\gamma}^a$, which is inserted into the cyclic product in front of the factor $\boldsymbol{g}_{-\lambda}$. In this case, the result is

$$\{u_\pi, z_\lambda^a\} = -2\pi G \, \mathrm{Tr}(\boldsymbol{\gamma}^a \prod_{\kappa \in \mathit{\Gamma}_{\pi,\Delta_\lambda}} \boldsymbol{g}_\kappa) = -2\pi G \, \mathrm{Tr}(\boldsymbol{\gamma}^a \, \boldsymbol{u}_{\pi,\Delta_\lambda}) = -16\pi^2 G^2 \, p^a_{\pi,\Delta_\lambda}. \tag{3.32}$$

All together, and again in matrix notation, we get

$$\{\mathcal{C}_\pi, \boldsymbol{z}_\lambda\} = \begin{cases} \boldsymbol{p}_{\pi,\Delta_\lambda} & \text{if } \pi = \pi_\lambda, \\ -\boldsymbol{p}_{\pi,\Delta_\lambda} & \text{if } \pi = \pi_{-\lambda}, \\ 0 & \text{otherwise.} \end{cases} \tag{3.33}$$

This is formally the same as (1.51), and it has the same interpretation. The mass shell constraint $\mathcal{C}_\pi$ generates the motion of the particle π along its world lines. The direction of this world line is specified by the momentum vector $\boldsymbol{p}_{\pi,\Delta}$ in the chart containing the polygon Δ. For $\Delta = \Delta_\lambda$, this is also the chart in which the vector $\boldsymbol{z}_\lambda$ is defined. The relative position vector $\boldsymbol{z}_\lambda$ sees the motion of the particles π_λ and $\pi_{-\lambda}$.

There is also a non-trivial action of the mass shell constraints on the clocks T_η. They have non-vanishing brackets with the energies M_η, and thus with the holonomies $\boldsymbol{g}_{\pm\eta}$ of the external links. It follows from (3.4) that for $\eta \in \mathit{\Gamma}_\infty$ and $-\eta \in \mathit{\Gamma}_{-\infty}$ we have

$$\{\boldsymbol{g}_\eta, T_\eta\} = -4\pi G \, \boldsymbol{g}_\eta \, \boldsymbol{\gamma}^0, \qquad \{\boldsymbol{g}_{-\eta}, T_\eta\} = 4\pi G \, \boldsymbol{\gamma}^0 \, \boldsymbol{g}_{-\eta}. \tag{3.34}$$

Formally, T_η acts on the transition functions $\boldsymbol{g}_{\pm\eta}$ as if it was the $\boldsymbol{\gamma}_0$-component of a fictitious relative position vector assigned to $-\eta$. Performing the same calculation again, it follows that

$$\{\mathcal{C}_\pi, T_\eta\} = \begin{cases} p^0_{\pi,\Delta_\eta} = p^0_{\pi,\Delta_{-\eta}} & \text{if } \pi = \pi_{-\eta}, \\ 0 & \text{otherwise.} \end{cases} \tag{3.35}$$

In contrast to (3.33), the case $\pi = \pi_\eta$ does not occur, because the external link η has no end point. And furthermore, we can here evaluate the momentum vector $\boldsymbol{p}_\pi$ in either of the two adjacent polygons Δ_η or $\Delta_{-\eta}$. The transition function between the two associated charts is $\boldsymbol{g}_\eta$, and this is a rotation about the $\boldsymbol{\gamma}_0$-axis. Consequently, the $\boldsymbol{\gamma}_0$-component of the vector $\boldsymbol{p}_\pi$ is the same in both adjacent polygons Δ_η and $\Delta_{-\eta}$.

Finally, if we ignore the dependence of the various vectors on the polygons, hence on the coordinate charts with respect to which they are defined, then the time evolution equations are the same as (1.52) for the free particles,

$$\dot{\boldsymbol{z}}_\lambda = \{\mathcal{H}, \boldsymbol{z}_\lambda\} = \zeta_{\pi_\lambda} \boldsymbol{p}_{\pi_\lambda} - \zeta_{\pi_{-\lambda}} \boldsymbol{p}_{\pi_{-\lambda}}, \qquad \dot{T}_\eta = \{\mathcal{H}, T_\eta\} = \zeta_{\pi_{-\eta}} p^0_{\pi_{-\eta}}. \tag{3.36}$$

And clearly, the physical interpretation of these equations is also the same as before. The particle π moves along its world line, by an amount that is specified by the multiplier ζ_π. Thereby, the geometry of the ADM surface changes, and also its embedding into the spinning cone. If all multipliers are positive, this finally provides a proper foliation of the spacetime manifold.

Symmetries

In the Hamiltonian formulation of general relativity, every isometry of the asymptotic metric at spatial infinity should be realized as a rigid symmetry at the phase space level [19]. Let us check this for the time translations $T \mapsto T - \Delta T$ and the spatial rotations $\phi \mapsto \phi + \Delta\phi$ of the spinning cone (2.15). And let us also derive the associated conserved charges. By definition, these are the total energy and the total angular momentum. First we have to find out how the symmetries act on the phase space variables. Then we can check whether these are symmetries of the symplectic structure, derive the charges, and finally we have to show that the charges commute with all constraints.

A time translation $T \mapsto T - \Delta T$ of the spinning cone does not affect the geometry of space. The whole ADM surface is just shifted backwards in time, with respect to the reference frame. Only the clocks T_η refer to the absolute time coordinate T of the spinning cone, so only they transform,

$$T_\eta \mapsto T_\eta - \Delta T. \tag{3.37}$$

Since ΔT is a constant, this is obviously a symmetry of the symplectic structure defined in (3.9). The symplectic potential changes by a total derivative $\Theta \mapsto \Theta - \Delta T\, \mathrm{d}M$, but the two-form $\Omega = \mathrm{d}\Theta$ is invariant. The associated charge is also easy to find. It is the previously defined total energy M, which has vanishing brackets with all phase space variables, except for the clocks,

$$M = \sum_{\eta \in \Gamma_\infty} M_\eta \quad \Rightarrow \quad \{M, T_\eta\} = -1. \tag{3.38}$$

It is also immediately obvious that M commutes with all kinematical and dynamical constraints, thus the total energy is in fact a conserved charge.

The same applies to a spatial rotation $\phi \mapsto \phi + \Delta\phi$ of the spinning cone. The local geometry of space is unchanged, while the whole ADM surface is rotated with respect to the spinning cone by an angle $\Delta\phi$ in conical coordinates. As only the angular directions ϕ_η of the external links refer to the conical coordinate ϕ, let us make the following ansatz,

$$\phi_\eta \mapsto \phi_\eta + \Delta\phi. \tag{3.39}$$

Now, suppose we apply this transformation. Then the deviations $\theta_\eta^\pm$ are invariant. They are implicitly defined in (3.6) and (3.7), and these equations are invariant under the given transformation. But if we then look at the definition (3.5) of the unit vectors $\boldsymbol{z}_{\pm\lambda}$, we find that these vectors are also rotated by an angle $\Delta\phi$, but now this rotation takes place in the auxiliary Minkowski space, where the polygons are embedded.

The reason can be seen in figure 5. If we rotate all external links simultaneously with respect to the spinning cone on the right, then we also have to rotate the external edges in the embedding Minkowski space on the left. This is because we required the overall, or average orientation of the two frames to coincide. But then, if we want the local geometry of the ADM surface to be unchanged, we also have to rotate the internal edges of all non-compact polygons. Otherwise the polygons are twisted, and the ADM surface is deformed. So, to define a proper spatial rotation of the particles with respect to the reference frame, we not only have the rotate the external links with respect to the spinning cone. We also have to rotate all non-compact polygons in the embedding Minkowski space by the same angle $\Delta\phi$.

But then it is actually simpler to rotate all polygons, including the compact ones. This just involves another kinematical gauge transformation. In other words, we have to apply the same rotation about the $\boldsymbol{\gamma}_0$-axis to all Minkowski charts, thus rotate all polygons in figure 3 by the same angle. And finally, we also have to adjust the transition functions. If we rotate the Minkowski coordinates in all charts, then we also have to apply this rotation to the transition functions. All together, we get the following transformation of the link variables, in addition to (3.39),

$$\boldsymbol{g}_\lambda \mapsto \mathrm{e}^{\Delta\phi\boldsymbol{\gamma}_0/2}\boldsymbol{g}_\lambda\,\mathrm{e}^{-\Delta\phi\boldsymbol{\gamma}_0/2}, \qquad \boldsymbol{z}_\lambda \mapsto \mathrm{e}^{\Delta\phi\boldsymbol{\gamma}_0/2}\boldsymbol{z}_\lambda\,\mathrm{e}^{-\Delta\phi\boldsymbol{\gamma}_0/2}, \qquad \lambda \in \mathit{\Gamma}. \tag{3.40}$$

For internal links, this defines the transformation of the phase space variables $\boldsymbol{z}_\lambda$ and $\boldsymbol{g}_\lambda$. For external links, it is consistent with the definitions (3.4) and (3.5), and the transformation (3.39).

Given these transformations, it is straightforward to check that the symplectic structure (3.9) is invariant. Thus we also have a symmetry of the phase space. To find the total angular momentum, let us first consider the internal links, and the definition (3.8) of the angular momenta L_λ. It is not difficult to derive the brackets

$$\{L_\lambda, \boldsymbol{z}_{\pm\lambda}\} = \tfrac{1}{2}[\boldsymbol{\gamma}_0, \boldsymbol{z}_{\pm\lambda}], \qquad \{L_\lambda, \boldsymbol{g}_{\pm\lambda}\} = \tfrac{1}{2}[\boldsymbol{\gamma}_0, \boldsymbol{g}_{\pm\lambda}], \qquad \lambda \in \mathit{\Gamma}_0. \tag{3.41}$$

The phase space function L_λ generates a counter clockwise rotation of the internal edges $\pm\lambda$ in the embedding Minkowski space. This was also the defining property of the angular momenta L_λ in (1.55), for the free particles. But now we also have to rotate the external links. The generator of the rotation (3.39) is the angular momentum L_η,

$$\{L_\eta, \phi_\eta\} = 1 \qquad \eta \in \mathit{\Gamma}_\infty. \tag{3.42}$$

To act on all links at the same time, we just have to sum over all internal and external links. Hence, the total angular momentum becomes

$$J = \sum_{\lambda \in \mathit{\Gamma}_+} L_\lambda + \sum_{\eta \in \mathit{\Gamma}_\infty} L_\eta. \tag{3.43}$$

This is formally the same as (1.58) for the free particles. To see that it is a conserved charge, we have to compute the brackets of J with the constraints. It turns out that the only non-vanishing brackets are those with the vector constraints for the compact polygons,

$$\{J, \boldsymbol{Z}_\Delta\} = \tfrac{1}{2}[\boldsymbol{\gamma}_0, \boldsymbol{Z}_\Delta] \approx 0, \qquad \Delta \in \mathit{\Pi}_0. \tag{3.44}$$

This is again proportional to the same constraint, so that J is at least weakly conserved. This is of course sufficient to define a conserved charge. We conclude that all symmetries of the spacetime are in fact realized as symmetries of the phase space.

But how is J related to the parameter S of the spinning cone? It is no longer so that J is weakly equal to S. Instead,

$$J = \sum_{\lambda \in \mathit{\Gamma}_+} L_\lambda + \sum_{\eta \in \mathit{\Gamma}_\infty} L_\eta \approx S - 4GS \sum_{\eta \in \mathit{\Gamma}_\infty} M_\eta = (1 - 4GM)\, S. \tag{3.45}$$

So, the total angular momentum is not the geometric parameter S, which defines the time offset of the spinning cone. It is rescaled by a factor $1 - 4GM$.

The definition of the total angular momentum is actually somewhat ambiguous. Of course, S is also a conserved charge. It is a function of J and M, at least up to terms proportional to the constraints. So, there is also a symmetry associated with S. It must be some combination of a time translation and a spatial rotation, thus a kind of *screw rotation*. And in fact, the definition of the rotational Killing vector of the spinning cone is also somewhat ambiguous. The Killing vector associated with time translations is uniquely defined as that Killing vector which is parallel to the fictitious world line of the centre of mass of the universe. Hence the definition of the total energy M is unique.

However, there are two alternative definitions of the rotational Killing vector, and both are quite natural. They coincide only if the spinning cone is actually static, thus if the angular momentum is zero. One possible definition is to choose the unique Killing vector which has closed orbits of affine length 2π. The associated angular momentum is then J, which has the property that a symmetry transformation with parameter $\Delta\phi = 2\pi$ is the identity. But we may also define the rotational Killing vector to be the unique Killing vector which is orthogonal to the Killing vector of time translations. On a spinning cone, this Killing vector generates a screw rotation, and it is this symmetry which is associated with the conserved charge S.

It is not clear which of the two is the *correct* angular momentum, S or J. Which one is the more appropriate one depends on the context. For example, if we quantize a particle model like this, then it is J whose eigenvalues are quantized in steps of $\hbar$, because the essential property is then the existence of closed orbits of affine length 2π. On the other hand, if we want to describe the conical geometry of the spacetime at infinity, then S is more natural, because it is this which appears in the metric (2.15) of the spinning cone, and which defines the time offset $8\pi GS$.

An interesting consequence of this interplay between quantization and geometry and the charges J and S is discussed in the very end of [21]. It turns out that, independent of the matter content of the universe, only those conical geometries can be realized in a quantized three dimensional universe, which fulfill the following Dirac like quantization condition,

$$\frac{(1-4GM)\,S}{\hbar} \in \mathbb{Z}. \tag{3.46}$$

Since here were are not going to quantize anything, we shall not go into any details and derive this quantization condition. But it is more or less obvious that this follows from the fact that J is quantized in integer multiples of $\hbar$. This result is independent of the non-commutative structure of spacetime implied by the brackets (3.13), but it also provides a kind of quantized geometry.

Parity transformations

Let us also briefly consider a discrete symmetry. Apparently, the Poisson algebra (3.13) is not invariant under parity transformations, thus under reflections of space, because the Levi Civita tensor shows up. But on the other hand we know that the Einstein equations are invariant, and thus for every spacetime there exists a reflected spacetime. It is therefore not immediately obvious that this symmetry is also realized at the phase space level. And in fact, for the previously considered single particle model this was not the case, due to a somewhat ambiguous definition of the reference frame [13].

For a single particle, we had the same Poisson algebra (3.13), but there was only one vector $\boldsymbol{z}$, representing the absolute position of the particle with respect to the reference frame. Now each vectors $\boldsymbol{z}_\lambda$ is the relative position vector of two particles, and there are several such vectors. But still, the Levi Civita tensor shows up, and we should therefore check whether the problem is still present or not. First of all, we have to find out how the various phase space variables actually transform under a reflection of space. We have to start at the very beginning, the definition of a triangulation. When we perform a reflection of space, we have to replace the graph Γ by its mirror image.

This is a non-trivial operation, because the cyclic orderings of the subsets Γ_π, Γ_∞ and Γ_Δ already refer to the orientation of space. For each particle π, we have to reverse the cyclic ordering of the set Γ_π of links ending at π, and the same applies to the set Γ_∞ of links pointing towards infinity. For each polygon Δ, we not only have to reverse the cyclic ordering of its edges in Γ_Δ, but also reverse the orientation of the edges. For example, if a polygon Δ is bounded by the edges $\Gamma_\Delta = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ before the parity transformation, then after the parity transformation we have $\Gamma_\Delta = \{-\lambda_n, \ldots, -\lambda_2, -\lambda_1\}$.

Now, consider the reflection of a polygon in the embedding Minkowski space, which is indicated in figure 7. The edge $-\lambda$ of the polygon $\Delta_{-\lambda}$ becomes, after the reflection, the edge λ of the polygon Δ_λ. The edge was originally represented by the vector $\boldsymbol{z}_{-\lambda}$. The polygon is then reflected, and additionally the orientation of the boundary is reversed. Hence, the new vector $\boldsymbol{z}_\lambda$ is obtained from the original vector $\boldsymbol{z}_{-\lambda}$ by a reflection and an overall change of sign. Let us choose the reflection in Minkowski space so that $\boldsymbol{\gamma}_2 \mapsto -\boldsymbol{\gamma}_2$, hence a reflection at the $\boldsymbol{\gamma}_0$-$\boldsymbol{\gamma}_1$-plane, which is given by (A.14)

Combining this with an overall change of sign, and deriving the appropriate transformation of

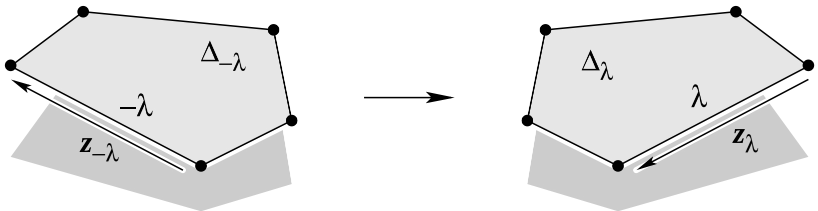

Figure 7: A parity transformation acts on a polygon in two ways. The polygon is first reflected, and then the orientation of the boundary is reversed. If the link $-\lambda$ is an edge of the original polygon $\Delta_{-\lambda}$, which is represented by the vector $\boldsymbol{z}_{-\lambda}$, then the reversed link λ becomes an edge of the reflected polygon Δ_{λ}, which is represented by the transformed vector $\boldsymbol{z}_{\lambda}$.

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the transition functions from (3.3) gives

$$\boldsymbol{z}_{\lambda} \mapsto \boldsymbol{\gamma}_2\, \boldsymbol{z}_{-\lambda}\, \boldsymbol{\gamma}_2, \qquad \boldsymbol{g}_{\lambda} \mapsto \boldsymbol{\gamma}_2\, \boldsymbol{g}_{-\lambda}\, \boldsymbol{\gamma}_2. \tag{3.47}$$

The external link variables also transform in a non-trivial way. It follows from the definition (3.8) of the angular momentum that $S \to -S$. Thus the reflected spinning cone is spinning into the opposite direction. The conical coordinates are reflected by the transformation $\phi \mapsto -\phi$. From this we infer that

$$T_{\eta} \mapsto T_{\eta}, \qquad \phi_{\eta} \mapsto -\phi_{\eta}, \qquad M_{\eta} \mapsto M_{\eta}, \qquad L_{\eta} \mapsto -L_{\eta}. \tag{3.48}$$

The last transformation follows from $S \mapsto -S$, and the constraint equation $\mathcal{J}_{\eta} \approx 0$ defined in (3.16), which must be satisfied before and after the reflection. The vector constraint $\boldsymbol{Z}_{\Delta} \approx 0$ in (3.14) is obviously also preserved. And for the constraint $\mathcal{Z}_{\Delta} \approx 0$ defined in (3.15), we have to take into account that the external links η and η' are interchanged, because the ordering of $\mathit{\Gamma}_{\infty}$ is reversed, so that this is also preserved.

The symplectic potential (3.9) can then be shown to be invariant. The external terms $T_{\eta}\,\mathrm{d}M_{\eta}$ and $L_{\eta}\,\mathrm{d}\phi_{\eta}$ are obviously invariant under the transformation (3.48). For the internal links, this is not immediately obvious. However, we have $\boldsymbol{\gamma}_2\boldsymbol{\gamma}_2 = \mathbf{1}$, and therefore

$$\mathrm{Tr}(\boldsymbol{g}_{\lambda}^{-1}\mathrm{d}\boldsymbol{g}_{\lambda}\, \boldsymbol{z}_{\lambda}) \mapsto \mathrm{Tr}(\boldsymbol{g}_{-\lambda}^{-1}\,\mathrm{d}\boldsymbol{g}_{-\lambda}\, \boldsymbol{z}_{-\lambda}). \tag{3.49}$$

But this just means that we have to replace $\mathit{\Gamma}_{+}$ by $\mathit{\Gamma}_{-}$ in (3.9). And since we know that the symplectic potential is independent of the decomposition $\mathit{\Gamma}_0 = \mathit{\Gamma}_{+} \cup \mathit{\Gamma}_{-}$, it follows that it is also invariant under parity transformations.

Finally, we also have to check the mass shell constraints, hence the traces of the holonomies u_{π} of the particles. In the definition (2.11), each factor $\boldsymbol{g}_{\lambda}$ in the product is replaced by $\boldsymbol{\gamma}_2\boldsymbol{g}_{-\lambda}\boldsymbol{\gamma}_2$, and the ordering of the factors is reversed. Again, the factors $\boldsymbol{\gamma}_2$ drop out because $\boldsymbol{\gamma}_2\boldsymbol{\gamma}_2 = \mathbf{1}$. So, the group element under the trace is effectively replaced by its inverse. But the trace of the inverse of an element of $\mathsf{SL}(2)$ is equal to the trace of the element itself.

So, we find that all the phase space structures are invariant under parity transformations. But what is then the difference to the previously considered single particle system, where this was not the case? The difference is that the definition of the relative position vector $\boldsymbol{z}_\lambda$ already refers to a fixed orientation of the spacetime. It defines the edge of a polygon, where the boundary is traversed in counter clockwise direction. This is why the Levi Civita tensor can show up in the Poisson algebra, without breaking the parity invariance.

Open and closed universes

So far we assumed that the topology of space is $\mathbb{R}^2$. But of course, general relativity allows a more general topology. The space manifold can be any two dimensional, orientable surface, and it can be either compact or non-compact. Let us for example consider a *closed* universe, where the space manifold is a Riemann surface of genus g. In this case, the graph $\mathit{\Gamma} = \mathit{\Gamma}_0$ consists of internal links only, and all polygons are compact, thus $\mathit{\Pi} = \mathit{\Pi}_0$. When the Riemann surface is triangulated, then we have the following relation between the number of vertices, links, polygons, and the genus,

$$\ell_0 - \wp_0 = n + 2g - 2. \tag{3.50}$$

We can use this to calculate the number of physical degree of freedom of a closed universe. The extended phase space $\mathcal{Q}_\mathit{\Gamma}$ is spanned by the independent link variables $\boldsymbol{g}_\lambda \in \mathsf{SL}(2)$ and $\boldsymbol{z}_\lambda \in \mathfrak{sl}(2)$ for $\lambda \in \mathit{\Gamma}_+$, hence

$$\dim(\mathcal{Q}_\mathit{\Gamma}) = 6\ell_0. \tag{3.51}$$

Then we have to impose the kinematical constraints. For a closed universe, we only have the vector constraints $\boldsymbol{Z}_\triangle$ for $\triangle \in \mathit{\Pi}_0$, and the associated gauge symmetries are the Lorentz rotation of the polygons. There is no relation like (3.17), and there is no restriction on the parameters of the Lorentz rotations. Thus we have $3\wp_0$ independent kinematical constraints, and also $3\wp_0$ independent gauge symmetries,

$$\dim(\mathcal{S}_\mathit{\Gamma}) = 6\ell_0 - 3\wp_0, \qquad \dim(\mathcal{S}_\mathit{\Gamma}/\sim) = 6\ell_0 - 6\wp_0 = 6n + 12g - 12. \tag{3.52}$$

This is the number of independent kinematical degrees of freedom, thus the phase space dimension before the mass shell constraints are imposed. To obtain the physical phase space, we have to impose n additional mass shell constraints, and subtract n additional dynamical gauge symmetries. Hence, the number of physical degrees of freedom is

$$\dim(\mathcal{P}_\mathit{\Gamma}) = 6\ell_0 - 3\wp_0 - n, \qquad \dim(\mathcal{P}_\mathit{\Gamma}/\sim) = 6\ell_0 - 6\wp_0 - 2n = 4n + 12g - 12. \tag{3.53}$$

For each particle we have 4 physical degrees of freedom. The particle is moving in a two dimensional space, thus we have two position and two momentum coordinates.

Additionally, the closed universe also has some topological degrees of freedom. Each *handle* of the Riemann surface contributes with 12 degrees of freedom. The topological degrees of freedom are the holonomies of the fundamental, non-contractible loops in space. There are two such loops for each handle, and regarding the holonomy we have to take into account both the rotational and the translational components [3, 4]. Hence, for each non-contractible loop we have one element of the six dimensional Poincaré group, and all together this makes $12g$ topological degrees of freedom, since the genus g counts the number of handles.

Finally, there is a consistency condition, which must be satisfied for the universe to be closed. This explains the minus twelve in the end. As an example, consider a sphere with $g = 0$. Then we have $4(n - 3)$ independent physical degrees of freedom. We need at least four particles to get a non-trivial dynamical system. In this case, the space at a moment of time looks like a tetrahedron, and we have four independent degrees of freedom. The relative motion of two particles already determines the motion of the other two particles. For higher genus surfaces, we need less particles or even no particles at all to get a non-trivial dynamical system.

For an *open* universe, we may also consider more general topologies. The space manifold can be represented as a *punctured* Riemann surface, thus a surface of genus g with $k \geq 1$ points taken away. Each puncture represents a region that extends to spatial infinity. For each such infinity, there is a conical frame, and a set of external links and non-compact polygons covering a neighbourhood of this particular infinity. Moreover, each conical frame has its own total energy and total angular momentum, and from each infinity an external observer can see the particles and also the other observers. The case that we discussed so far, and which reduces to the free particle system in the limit $G \to 0$, is the special open universe with $g = 0$ and $k = 1$. The closed universes are also special cases, with $k = 0$.

Let us count the number of physical degrees of freedom for an open universe. First we have to derive the relation between the number of vertices, links, and polygons. It is the same as (3.50), but now we have a vertex not only for every particle but also for every infinity. Moreover, we have to distinguish between internal and external links. The external links are now those that extend to any of the k infinities. But there are still as many external links are there are non-compact polygons. Hence,

$$\ell_0 - \wp_0 = n + k + 2g - 2, \qquad \ell_\infty = \wp_\infty. \tag{3.54}$$

The phase space dimension is counted in the same way as before. The extended phase space $\mathcal{Q}_\Gamma$ is spanned by $6\ell_0$ independent components of the group elements $\boldsymbol{g}_\lambda$ and the vectors $\boldsymbol{z}_\lambda$ for $\lambda \in \Gamma_0$, and $4\ell_\infty$ real variables M_η, L_η, T_η, ϕ_η for $\eta \in \Gamma_\infty$, thus

$$\dim(\mathcal{Q}_\Gamma) = 6\ell_0 + 4\ell_\infty. \tag{3.55}$$

Then we have to count the number of kinematical constraints. There are $3\wp_0$ components of the vector constraints $\boldsymbol{Z}_\triangle$ for $\triangle \in \Pi_0$, $\wp_\infty$ scalar constraints $\mathcal{Z}_\triangle$ for $\triangle \in \Pi_\infty$, and finally ℓ_∞ scalar constraints $\mathcal{J}_\eta$ for $\eta \in \Gamma_\infty$. But now they are not all independent. For each infinity, we have a relation like (3.17). This implies that the total number of independent kinematical constraints is $3\wp_0 + \wp_\infty + \ell_\infty - k$, and this is also the number of independent redundancy transformations. It follows that

$$\begin{aligned} \dim(\mathcal{S}_\Gamma) &= 6\ell_0 - 3\wp_0 + 2\ell_\infty + k, \\ \dim(\mathcal{S}_\Gamma/\!\sim) &= 6\ell_0 - 6\wp_0 + 2k = 6n + 12g + 8k - 12. \end{aligned} \tag{3.56}$$

This is the number of independent kinematical degrees of freedom. To get the physical degrees of freedom, we have to impose n additional mass shell constraints, and subtract another n gauge symmetries. This gives

$$\begin{aligned} \dim(\mathcal{P}_\Gamma) &= 6\ell_0 - 3\wp_0 + 2\ell_\infty - n + k, \\ \dim(\mathcal{P}_\Gamma/\!\sim) &= 6\ell_0 - 6\wp_0 - 2n + 2k = 4n + 12g + 8k - 12. \end{aligned} \tag{3.57}$$

Once again, we have four degrees of freedom for every particle, and twelve degrees of freedom for every handle. And obviously there are eight degrees of freedom associated with every infinity. This can be understood as follows. There are two charges M and S, defining the conical geometry, which can be different at each infinity. The other six degrees of freedom represent the relative orientation of the conical frames with respect to each other.

Consider for example two infinities and the associated conical frames. Think of them as the rest frames of two different observers. When the observers talk to each other, they find out that they do not agree about the fictitious position of the centre of mass of the universe, and in general they are also in relative motion with respect to each other. Hence, every infinity has its own centre of mass frame. The relation between two such frames is, at least locally in phase space, parameterized by a Poincaré transformation. If we fix one reference frame, then for every other there are six degrees of freedom, representing the relative position and orientation with respect to the fixed one.

The minus twelve can now be explained by looking at the special case $k = 1$ and $g = 0$, which reduces to the free particle system in the limit $G \to 0$. In this case, the dimension of the reduced phase space is $4(n-1)$. This is the number of physical degrees of freedom of the relative motion of n particles in a two dimensional space. To get a non-trivial dynamical system, we need at least two particles. In general, each additional infinity, and also each handle already provides sufficiently many topological degrees of freedom, so that we do not need any particles at all to obtain a non-trivial dynamical system. These are the usual, say, cosmological toy models, where only the topological degrees of freedom are present, and they were the first quantized toy models in three dimensional gravity [3].

Large gauge symmetries

Finally, let us also have a look at the global structure of the phase space. We have seen that the extended phase space $\mathcal{Q}$ consists of finitely many disconnected components $\mathcal{Q}_\Gamma$, one for each possible graph Γ with n particles. According to (3.55), they even have different dimensions. It is therefore not possible to consider the components $\mathcal{Q}_\Gamma$ of the extended phase space $\mathcal{Q}$ as overlapping coordinate charts of a single manifold. However, we can still glue them together in a certain way, so that the quotient spaces $\mathcal{S}/{\sim}$ and $\mathcal{P}/{\sim}$ become connected manifold. So far, we only considered the gauge symmetries generated by the constraints.

Two states $\Phi_1, \Phi_2 \in \mathcal{S}_\Gamma$, belonging to the same component of the kinematical subspace, are defined to be physically equivalent, $\Phi_1 \sim \Phi_2$, if they can be smoothly deformed into each other by a transformation which is generated by the kinematical constraints. This is equivalent to the condition that they lie on the same *gauge orbit*, that is the same null orbit of the symplectic structure on $\mathcal{S}_\Gamma$. However, there are also states that belong to different components of $\mathcal{S}$, but which are nevertheless equivalent because they define the same geometry of space. They are then related by a *large* gauge transformation. In the end of section 2, we have seen how these transformations look like. They can be decomposed into elementary steps, namely inserting and removing links.

From the phase space point of view, the typical situation is shown in figure 8. There we see three components of the extended phase space $\mathcal{Q}$, corresponding to the graphs $\Gamma^{(0)}$, $\Gamma^{(1)}$, and $\Gamma^{(2)}$. Let us assume that both $\Gamma^{(1)}$ and $\Gamma^{(2)}$ are obtained from $\Gamma^{(0)}$ by inserting a new pair of

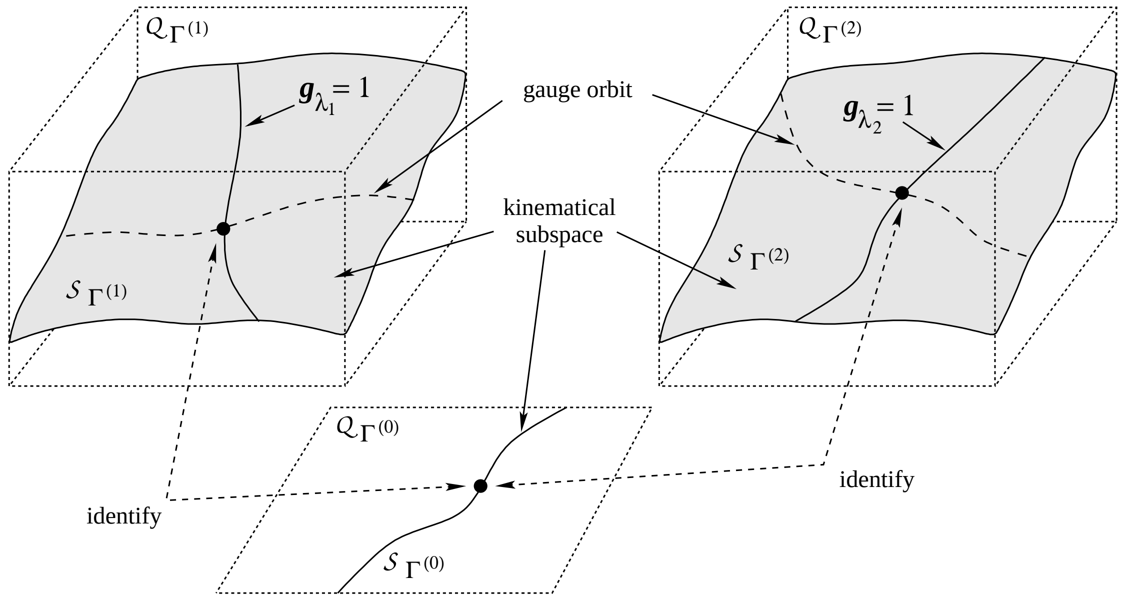

Figure 8: The extended phase space $\mathcal{Q}$ consists of finitely many disconnected components $\mathcal{Q}_\Gamma$ of different dimension. A large gauge symmetry embeds a lower dimensional component $\mathcal{S}_\Gamma$ of the kinematical phase space into the respective higher dimensional component. The embedding is given by the insertion of a new link λ, as shown in figure 6. The image of the lower dimensional component in the higher dimensional one is the subspace defined by the gauge condition $\boldsymbol{g}_\lambda = \mathbf{1}$.

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links, $\pm\lambda_1$ and $\pm\lambda_2$, respectively. For example, the new links may be two different diagonals of the polygon in figure 6. The component $\mathcal{Q}_{\Gamma^{(0)}}$ is the lower dimensional one, and the components $\mathcal{Q}_{\Gamma^{(1)}}$ and $\mathcal{Q}_{\Gamma^{(2)}}$ have the same higher dimension. The same applies to the kinematical subspaces $\mathcal{S}_{\Gamma^{(0)}}$, $\mathcal{S}_{\Gamma^{(1)}}$, and $\mathcal{S}_{\Gamma^{(2)}}$, which are submanifolds of the respective components of $\mathcal{Q}$.

The transition from the graph $\Gamma^{(0)}$ to $\Gamma^{(1)}$ is a large gauge symmetry, which maps a state $\Phi^{(0)} \in \mathcal{S}_{\Gamma^{(0)}}$ onto an equivalent state $\Phi^{(1)} \in \mathcal{S}_{\Gamma^{(1)}}$. And a transition from $\Gamma^{(0)}$ to $\Gamma^{(2)}$ maps the state $\Phi^{(0)} \in \mathcal{S}_{\Gamma^{(0)}}$ onto an equivalent state $\Phi^{(2)} \in \mathcal{S}_{\Gamma^{(2)}}$. We can think of an *embedding* of the lower dimensional component $\mathcal{S}_{\Gamma^{(0)}}$ into the higher dimensional components $\mathcal{S}_{\Gamma^{(1)}}$ and $\mathcal{S}_{\Gamma^{(2)}}$. The image of this embedding is a submanifold of the higher dimensional component of the kinematical phase space. It is the submanifold which is defined by the *gauge condition* $\boldsymbol{g}_{\lambda_1} = \mathbf{1}$ in $\mathcal{S}_{\Gamma^{(1)}}$, and $\boldsymbol{g}_{\lambda_2} = \mathbf{1}$ in $\mathcal{S}_{\Gamma^{(2)}}$, respectively.

A general kinematical gauge transformation can be described as follows. As long as we stick to a given triangulation, say $\Gamma^{(1)}$, the gauge symmetries are generated by the kinematical constraints. We follow a gauge orbit, hence a null direction of the symplectic structure on the kinematical subspace $\mathcal{S}_{\Gamma^{(1)}}$. Now, suppose that we want to remove the pair of links $\pm\lambda_1$. Then, we first have to fix a gauge where $\boldsymbol{g}_{\lambda_1} = \mathbf{1}$. Thus we have to follow the gauge orbit to some state where this gauge condition is satisfied. This state $\Phi^{(1)} \in \mathcal{S}_{\Gamma^{(1)}}$ is the image of some state $\Phi^{(0)} \in \mathcal{S}_{\Gamma^{(0)}}$ in the lower dimensional component, and the large gauge transformation takes us

there.

If we want, we can then perform another large gauge transformation, inserting a new link λ_2. Then we have a state $\Phi^{(2)}$ in the component $\mathcal{S}_{\Gamma^{(2)}}$ of the kinematical phase space. There we can again perform a smoothly generated gauge transformation, following a gauge orbit provided by the kinematical constraints, and so on. All components of the phase space are finally connected in this way, and consequently the quotient space $\mathcal{S}/\!\sim$ becomes a connected manifold. That it is a proper manifold follows from the fact that all components $\mathcal{S}_{\Gamma}/\!\sim$ of this quotient space have the same dimension. According to (3.56), this dimension only depends on the number of particles and the topology, but not on the triangulation.

However, what we still have to show is that $\mathcal{S}/\!\sim$ is in fact a proper symplectic manifold, and that the Hamiltonian, or the mass shell constraints are well defined functions thereon. In other words, we have to show that the symplectic structure is invariant under large gauge symmetries. This is the case if the symplectic potential on the lower dimensional component $\mathcal{S}_{\Gamma^{(0)}}$ is equal to the one that is induced by the embedding into the higher dimensional components $\mathcal{S}_{\Gamma^{(1)}}$ and $\mathcal{S}_{\Gamma^{(2)}}$. And the same has to apply to the holonomies of the particles. It is not difficult to see that this is indeed the case. If the inserted link is an internal link λ, then the image of the lower dimensional component in the higher dimensional one is the subspace defined by the gauge condition $\boldsymbol{g}_{\lambda} = \mathbf{1}$. If we look at the symplectic potential (3.9), we find that on this subspace the terms involving the link λ are just absent.

Hence, the symplectic potential is the same as if the link was not there. The same holds if the inserted link is an external link η. In this case, the image of the lower dimensional component in the higher dimensional one is the subset defined by $M_{\eta} = 0$, and consequently also $L_{\eta} = 0$, which follows from the constraint (3.16). Again, this implies that the terms involving the link η are not present in the expression (3.9) for the symplectic potential. Since we also know that no other link variables are affected when we insert a new link, we conclude that the symplectic structure on the lower dimensional component of the kinematical phase space is equal to that on its image in the higher dimensional component.

The same holds for the mass shell constraints and the Hamiltonian, which is given by (3.29). If any of the transition functions is trivial, thus $\boldsymbol{g}_{\lambda} = \mathbf{1}$, then it does not contribute to the holonomies in (3.25). Once again, it is therefore irrelevant for the definition of the holonomy whether the new link has been inserted or not. And the value of the Hamiltonian is consequently also invariant under large gauge symmetries. All together, this implies that the quotient space $\mathcal{S}/\!\sim$ is a well defined symplectic manifold, and the Hamiltonian, which is actually a linear combination of the mass shell constraints, is a well defined function thereon.

For the free particle system, we had an explicit definition of this manifold. It was the original, $6n-4$ dimensional phase space spanned by the positions $\boldsymbol{x}_{\pi}$ and the momenta $\boldsymbol{p}_{\pi}$, with the restriction to the centre of mass frame imposed. Here we do not have such an explicit definition. All we have is a finite atlas, where each chart is a labeled by a graph Γ, and where the coordinates are equivalence classes of kinematical states. Hence, both at the spacetime level and at the phase space level we only have local coordinates. A global chart on the phase space can only be defined if we also introduce a global coordinate chart on the spacetime manifold [14, 15]. But then we also have to give up the simple physical interpretation of the phase space coordinates. It is not possible to use the relative position vectors and the conjugate transition functions as global coordinates on the phase space [18].

4 Einstein gravity

In this section we are going to derive the phase space structures defined in section 3 from the Einstein Hilbert action. The starting point is the definition of the particle model at the level of general relativity as a field theory on a fixed spacetime manifold $\mathcal{M}$. It is convenient to stick to the ADM formulation from the very beginning, so that $\mathcal{M} = \mathbb{R} \times \mathcal{N}$ is foliated by a space manifold $\mathcal{N}$ evolving with respect to some unphysical ADM time coordinate t. Generalizing the definition of the previously considered single particle model, we define $\mathcal{N}$ to be a two dimensional surface, which is either a plane or a sphere, and we cut out with n open discs, so that n circular boundaries arise, representing the locations of the particles in space.

This provides a proper regularization of the conical singularities, and it formally converts the matter degrees of freedom associated with the particles into topological degrees of freedom of the gravitational field. A set of boundary conditions is imposed to make the boundaries look like points in space, and boundary terms are added to the action, to define the coupling of the particles to the gravitational field. A comprehensive description can be found in section 2 of [13]. For an open universe, we finally impose a fall off condition at spatial infinity. This is a kind of asymptotical flatness condition. It defines the centre of mass frame of the universe as a reference frame in the sense of [19].

To the so defined model we shall then apply the phase space reduction. We set up the Hamiltonian framework, identify the phase space, derive and solve the constraints, and finally the gauge degrees of freedom can be divided out. We'll find a finite number of *observables*, representing the physical degrees of freedom of the model, and these are the link variables used in figure 3 to define the phase space. We shall also recover the symplectic structure and the various kinematical constraints defined in the beginning of section 3, which are then shown to be derived from the Einstein Hilbert action.

The space manifold and the particles

The ADM space manifold $\mathcal{N}$ is a two dimensional, orientable surface with n circular boundaries, representing the locations of the particles in space. The global topology of $\mathcal{N}$ is in principle arbitrary. But for simplicity, and also for the reasons given in section 3, we shall here restrict to two special cases. For an *open* universe, the space manifold $\mathcal{N}$ is a plane $\mathbb{R}^2$ with n open discs cut out, as shown in figure 9. For a *closed* universe, the same n open discs are cut out from a sphere S^2.

In the first order dreibein formulation of general relativity, the basic field variables are the dreibein $\boldsymbol{e}_\mu$ and the spin connection $\boldsymbol{\omega}_\mu$, where $\mu, \nu, \ldots$ are formal tangent indices on $\mathcal{M} = \mathbb{R} \times \mathcal{N}$. For both open and closed universes, the tangent bundle of the spacetime manifold is trivial. Both fields are therefore $\mathfrak{sl}(2)$-valued one-forms. We don't have to define them as sections in a bundle. The metric and the dreibein determinant are given by

$$g_{\mu\nu} = \tfrac{1}{2}\mathrm{Tr}(\boldsymbol{e}_\mu \boldsymbol{e}_\nu), \qquad e = \tfrac{1}{12}\varepsilon^{\mu\nu\rho}\,\mathrm{Tr}(\boldsymbol{e}_\mu \boldsymbol{e}_\nu \boldsymbol{e}_\rho) > 0, \tag{4.1}$$

where $\varepsilon^{\mu\nu\rho}$ is the Levi Civita tensor on $\mathcal{M}$. We require the metric to be invertible, and the determinant to be positive everywhere in the *interior* of $\mathcal{M}$. What happens at the boundaries will be explained in a moment.

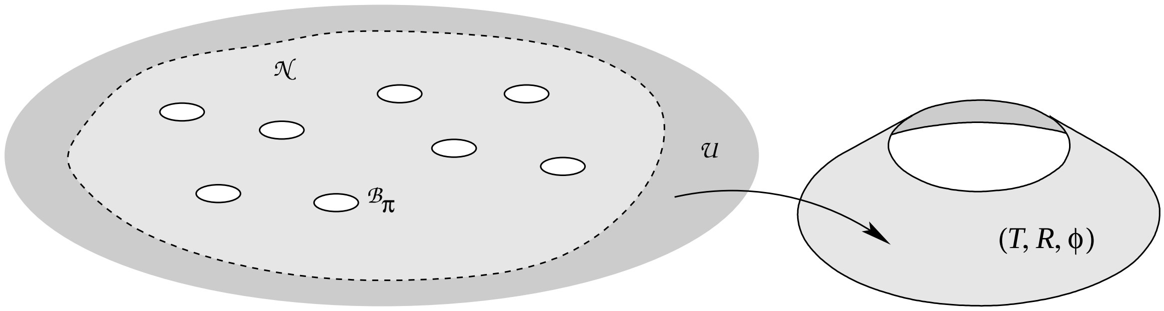

Figure 9: The space manifold $\mathcal{N}$ for an open universe is a plane $\mathbb{R}^2$ with n open discs cut out. For a closed universe, the same open discs are cut out from a sphere S^2. The circular boundary $\mathcal{B}_\pi$ represents the location of the particle π in space. The metric is degenerate on the boundaries, so that they look like points in space. The fall off condition at infinity requires the existence of a neighbourhood of infinity $\mathcal{U} \subset \mathcal{N}$, which is embedded into a spinning cone. This defines the centre of mass frame and serves as a reference frame.

In the ADM framework, the fields are decomposed into one-forms $\boldsymbol{e}_i$ and $\boldsymbol{\omega}_i$ on $\mathcal{N}$, where $i, j, \ldots$ are formal tangent indices, and scalars $\boldsymbol{e}_t$ and $\boldsymbol{\omega}_t$. The induced metric on $\mathcal{N}$ and the normal vector are given by

$$g_{ij} = \tfrac{1}{2}\mathrm{Tr}(\boldsymbol{e}_i\boldsymbol{e}_j), \qquad \boldsymbol{n} = \tfrac{1}{2}\varepsilon^{ij}\,[\boldsymbol{e}_i, \boldsymbol{e}_j]. \tag{4.2}$$

For the Levi Civita tensor on $\mathcal{N}$ we have $\varepsilon^{ij} = \varepsilon^{tij}$. For the foliation to by spacelike, and the time orientation of the dreibein to coincide with that of the ADM time coordinate t, the normal vector $\boldsymbol{n}$ is required to be *negative timelike*. A future pointing timelike vector ξ^μ is then mapped onto a positive timelike Minkowski vector $\xi^\mu \boldsymbol{e}_\mu$ by the dreibein.

The *particle boundaries* of $\mathcal{N}$ are denoted by $\mathcal{B}_\pi$. For a circular boundary to look like a point in space, its circumference has to vanish. This is the case if and only if the tangent component of the dreibein vanishes on the boundary. Let (r, φ) by a polar coordinate system in the neighbourhood of the boundary, so that the boundary is at $r = 0$, and φ increases in counter clockwise direction with a period of 2π. The *point particle condition* then requires

$$\boldsymbol{e}_\varphi|_{\mathcal{B}_\pi} = 0. \tag{4.3}$$

This implies that both the normal vector $\boldsymbol{n}$ and the dreibein determinant e are zero on the boundary. Therefore, the restriction to negative timelike normal vectors and invertible metrics applies in the interior of $\mathcal{N}$ only. For a one-dimensional boundary of the space manifold to look like a point, the metric on the boundary has to be degenerate.

From the spacetime point of view, the two-dimensional cylindrical boundary $\mathbb{R} \times \mathcal{B}_\pi$ represents a one-dimensional object, the world line of the particle π, and therefore the spacetime metric on the boundary has to be degenerate as well. But this is not going to be a problem, because in the first order formulation of Einstein gravity in three dimensions, the inverse dreibein never shows up. Blowing up the world line to a cylindrical boundary in fact removes all the singularities,

which are otherwise present in the dreibein and the spin connection describing the gravitational field of a point particle [13].

Action and boundary terms

The next step is to define the action functional for the particle model. In the first order dreibein formulation, the ADM Lagrangian depends on the fields $\boldsymbol{e}_\mu$ and $\boldsymbol{\omega}_\mu$ and their time derivatives. It is the sum of a bulk term, a boundary term for each particle boundary, and for an open universe there is also a boundary term at spatial infinity,

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_\infty - \sum_\pi \zeta_\pi \, \mathcal{C}_\pi. \tag{4.4}$$

The bulk term $\mathcal{L}_0$ is the usual first order Einstein Hilbert Lagrangian, which is given by

$$\mathcal{L}_0 = \frac{1}{8\pi G} \int_{\mathcal{N}} \mathrm{d}^2 x \, \varepsilon^{ij} \, \mathrm{Tr}(\dot{\boldsymbol{\omega}}_i \, \boldsymbol{e}_j + \tfrac{1}{2} \boldsymbol{\omega}_t \, \boldsymbol{T}_{ij} + \tfrac{1}{2} \boldsymbol{e}_t \, \boldsymbol{F}_{ij}). \tag{4.5}$$

The dot denotes the derivative with respect to t, and $\boldsymbol{F}_{ij}$ and $\boldsymbol{T}_{ij}$ are the spatial components of the curvature and the torsion, respectively,

$$\begin{aligned} \boldsymbol{F}_{\mu\nu} &= \partial_\mu \boldsymbol{\omega}_\nu - \partial_\nu \boldsymbol{\omega}_\mu + [\boldsymbol{\omega}_\mu, \boldsymbol{\omega}_\nu], \\ \boldsymbol{T}_{\mu\nu} &= \partial_\mu \boldsymbol{e}_\nu - \partial_\nu \boldsymbol{e}_\mu + [\boldsymbol{\omega}_\mu, \boldsymbol{e}_\nu] - [\boldsymbol{\omega}_\nu, \boldsymbol{e}_\mu]. \end{aligned} \tag{4.6}$$

This is the same as (3.1) in [13], with Newton's constant restored. It is the standard form of the Einstein Hilbert Lagrangian in three spacetime dimensions. The variation of the bulk term with respect to the dreibein and the spin connection, and with all boundary terms neglected, implies that

$$\boldsymbol{F}_{\mu\nu} = 0, \qquad \boldsymbol{T}_{\mu\nu} = 0. \tag{4.7}$$

These are the vacuum Einstein equations, stating that the spacetime is locally flat outside the matter sources.

The coupling of the particles to the gravitational field is introduced as follows. At each particle boundary $\mathcal{B}_\pi$, we add a *mass shell constraint* $\mathcal{C}_\pi$ to the Lagrangian, with a multiplier ζ_π as a coefficient. Formally, the mass shell constraints are given by (3.29),

$$\mathcal{C}_\pi = \frac{u_\pi - \cos(4\pi G m_\pi)}{16\pi^2 G^2} \tag{4.8}$$

But the holonomy of the particle is now defined as a function of the spin connection on the boundary,

$$u_\pi = \tfrac{1}{2} \mathrm{Tr}(\mathcal{P}\mathrm{exp} \int_{\mathcal{B}_\pi} \mathrm{d}\varphi \, \boldsymbol{\omega}_\varphi). \tag{4.9}$$

The path ordered exponential under the trace represents the Lorentz rotation acting on a vector which is transported once around the particle.

So far, this is a straightforward generalization of the previously defined single particle model. The mass shell constraint serves in at this level a two-fold purpose. It depends on the spin connection and therefore provides a boundary term for the Lagrangian. One can show that this,

together with the point particle condition and the vacuum Einstein equations, implies that the particle is moving on a geodesic, and that there is a conical singularity on this geodesic. On the other hand, the actually mass shell constraint, which is the equation of motion for the multiplier ζ_π, implies that this geodesic is timelike if the particle is massive and lightlike if it is massless, and it fixed the deficit angle of the conical singularity. All this is shown in [13], and it now applies to each particle independently.

Asymptotical flatness

For a closed universe, we now already have a well defined dynamical system. The configuration space is spanned by the fields $\boldsymbol{e}_\mu$ and $\boldsymbol{\omega}_\mu$, and the multipliers ζ_π. The fields are defined on a fixed space manifold $\mathcal{N}$, and they are subject to various boundary conditions. We have a well defined Lagrangian, which depends on the configuration variables and their time derivatives. The boundary term $\mathcal{L}_\infty$ is absent for a closed universe, and the space manifold $\mathcal{N}$ is compact, so that the integral defining the bulk term (4.5) is finite.

For an open universe, this is not the case. We have to impose some fall off conditions at infinity, and we also have to add an appropriate boundary term $\mathcal{L}_\infty$. Let us require that, far away from the particles, the spacetime looks like the gravitational field of a single particle, whose mass M is the total energy, and whose spin S is the total angular momentum of the universe. The metric at infinity is then that of a *spinning cone*, which we already encountered in section 2,

$$\mathrm{d}s^2 = -(\mathrm{d}T - 4GS\,\mathrm{d}\phi)^2 + \mathrm{d}R^2 + (1-4GM)^2\,R^2\,\mathrm{d}\phi^2. \tag{4.10}$$

Due to the absence of physical degrees of freedom, or local excitations of the gravitational field, we do not have to think about the actual fall off behaviour of the metric at infinity.

The spinning cone is, up to coordinate transformations, the most general solution to the vacuum Einstein equations in the considered region of spacetime, which admits the definition of a centre of mass frame. The condition for a centre of mass frame to exist is the same as for the free particle system. The total momentum must be timelike. For the gravitating particles, this total momentum is actually the total holonomy. It is defined like the holonomy of the particles, but with the spin connection integrated along the circle at infinity. If the total holonomy is not timelike, then the universe does not even even admit a proper causal structure [7, 8]. On the other hand, if it is timelike, then it is always possible to choose coordinates so that the metric becomes (4.10) [4, 5].

The region where the spinning cone metric applies is a *neighbourhood of infinity*. A neighbourhood of infinity in space is a subset $\mathcal{U} \subset \mathcal{N}$, which has the topology of $\mathbb{R}_+ \times \mathsf{S}^1$, and whose complement is compact. A typical neighbourhood of infinity is shown in figure 9. A neighbourhood of infinity in spacetime is a subset of $\mathcal{M} = \mathbb{R} \times \mathcal{N}$, so that at each moment of time the intersection with the space manifold $\mathcal{N}$ at that moment of time is a neighbourhood of infinity in space. So, the fall off condition to be imposed on the metric is that, at each moment of ADM time t, there exists a neighbourhood of infinity $\mathcal{U}$, and in $\mathcal{U}$ it is possible to introduce a conical coordinate system (T, R, ϕ) so that (4.10) holds.

Let us translate this into a fall off condition for the dreibein and the spin connection. A possible representation of the spinning cone metric is given by

$$\boldsymbol{e}_\mu = (\partial_\mu T + 4GS\,\partial_\mu\phi)\,\boldsymbol{\gamma}_0 + \partial_\mu R\,\boldsymbol{\gamma}(\phi) + (1-4GM)\,R\,\partial_\mu\phi\,\boldsymbol{\gamma}'(\phi),$$

$$\boldsymbol{\omega}_\mu = -2GM\, \partial_\mu \phi\, \boldsymbol{\gamma}_0. \tag{4.11}$$

One can easily check that this implies $\boldsymbol{F}_{\mu\nu} = 0$ and $\boldsymbol{T}_{\mu\nu} = 0$, so that the spinning cone is indeed a solution to the vacuum Einstein equations. However, it is not possible to impose this directly as a boundary condition on the basic field variables, because it still involves time derivatives. But it turns out to be sufficient to impose only the spatial components of (4.11) as boundary conditions at infinity.

The precise fall off condition is given as follows. At each moment of ADM time t, there exists a neighbourhood of infinity $\mathcal{U} \subset \mathcal{N}$, a pair of real numbers M and S, and a set of conical coordinates (T, R, ϕ) on $\mathcal{U}$, so that

$$\begin{aligned} \boldsymbol{e}_i &= (\partial_i T + 4GS\, \partial_i \phi)\, \boldsymbol{\gamma}_0 + \partial_i R\, \boldsymbol{\gamma}(\phi) + (1 - 4GM)\, R\, \partial_i \phi\, \boldsymbol{\gamma}'(\phi), \\ \boldsymbol{\omega}_i &= -2GM\, \partial_i \phi\, \boldsymbol{\gamma}_0. \end{aligned} \tag{4.12}$$

Note that we do not fix the conical coordinates in any way, or require the conical time T to be related in any way to the ADM time T. The conical coordinates are instead considered as additional field variables supported on $\mathcal{U}$, which are related to the dreibein and the spin connection by this fall off condition. And it is also useful to note that M and S are implicitly defined as functions of the spatial components of the dreibein and the spin connection.

Now, suppose that the dreibein and the spin connection satisfy this condition at each moment of time. Is the bulk term (4.5) of the Lagrangian then well defined? The last two terms are finite. The torsion $\boldsymbol{T}_{ij}$ and the curvature $\boldsymbol{F}_{ij}$ both vanish on $\mathcal{U}$, and therefore these terms have a compact support. What remains is the first term, involving the time derivative. In $\mathcal{U}$, it is given by

$$\frac{1}{8\pi G}\, \varepsilon^{ij}\, \mathrm{Tr}(\dot{\boldsymbol{\omega}}_i\, \boldsymbol{e}_j) = \frac{M}{2\pi}\, \varepsilon^{ij}\, \partial_i[(\dot{T} + 4GS\, \dot{\phi})\, \partial_j \phi]. \tag{4.13}$$

Here we added a total time derivative, which simplifies the result slightly. This is of course allowed, as we may at any time add a total time derivative to the Lagrangian. So, within the neighbourhood of infinity $\mathcal{U}$, the integrand in (4.5) can obviously be written as total spatial derivative. To obtain a well defined finite Lagrangian, we just have to subtract the resulting boundary term at infinity. Hence, the appropriate term $\mathcal{L}_\infty$ to be added in (4.4) is

$$\mathcal{L}_\infty = -\frac{M}{2\pi} \int_\infty \mathrm{d}s\, \partial_s \phi\, (\dot{T} + 4GS\, \dot{\phi}). \tag{4.14}$$

The notation means that the integral is evaluated along a circle at infinity, in counter clockwise direction. More precisely, the total Lagrangian is defined as follows. We first integrate (4.5) over a connected compact subset $\mathcal{N}_0 \subset \mathcal{N}$, which is sufficiently large, so that $\mathcal{N} = \mathcal{N}_0 \cup \mathcal{U}$, and so that the boundary $\partial\mathcal{N}_0$ is a loop in $\mathcal{U}$. Then we add the term (4.14), integrated along the boundary $\partial\mathcal{N}_0$. According to (4.13), the result is invariant under deformations of the boundary $\partial\mathcal{N}_0$. Thus we may finally also take the limit $\mathcal{N}_0 \to \mathcal{N}$, where the boundary $\partial\mathcal{N}_0$ becomes a circle at infinity.

With the fall off condition imposed and the boundary term added, the Lagrangian becomes finite. However, it is no longer only a function of the dreibein and the spin connection. The conical coordinates and their time derivatives show up explicitly in the boundary term. We have to include them into the definition of the configuration space. For a given geometry of the ADM surface, the conical coordinates are only fixed up to a time translation $T \mapsto T - \Delta T$, and a spatial

rotation $\phi \mapsto \phi + \Delta\phi$. These are the Killing symmetries of the spinning cone. So, we have to add two extra degrees of freedom to the configuration space, in order to obtain a well defined Lagrangian.

This is a typical feature of general relativity in the Lagrangian, or Hamiltonian framework [19]. The additional degrees of freedom represent a *reference frame*, and the Killing symmetries are the possible translations and rotation of the rest of the universe, thus in this case the particles, with respect to this reference frame. By definition, this reference frame coincides with the centre of mass frame of the universe. It is this feature which is responsible for the various ambiguities associated with the reference frame to disappear, when we go over from a single to a multi particle model. In [13], we found that the boundary term at infinity is not uniquely determined by the condition that the Lagrangian should be finite.

To be precise, this is also not the case here. We are still free to add any function of M and S to the Lagrangian. However, one can show that the given boundary term is uniquely fixed by the addition condition that the reference frame must not be translated or rotated when time evolves. If we derive the equations of motion and take into account that the conical coordinates are now also configuration variables, we find in addition to the vacuum Einstein equations the following time evolution equations for the conical coordinates,

$$\begin{aligned} \boldsymbol{e}_t &= (\dot{T} + 4GS\,\dot{\phi})\,\boldsymbol{\gamma}_0 + \dot{R}\,\boldsymbol{\gamma}(\phi) + (1 - 4GM)\,R\,\dot{\phi}\,\boldsymbol{\gamma}'(\phi), \\ \boldsymbol{\omega}_t &= -2GM\,\dot{\phi}\,\boldsymbol{\gamma}_0. \end{aligned} \tag{4.15}$$

Clearly, these are just the time components of (4.11). If we would add a function of M or S to the Lagrangian, then there would be additional time translation or rotations of the conical coordinates modifying the equations (4.15). Physically, this would mean that the reference frame is translated or rotated while the universe evolves, and this is of course not what we want. We therefore conclude that the Lagrangian is uniquely fixed by the condition that it must be finite, and that it has to provide the correct equations of motion for the reference frame.

Hamiltonian formulation

Let us now go over from the Lagrangian to the Hamiltonian formulation. This is actually only a formal step, because the Lagrangian is already of first order in time derivatives. The phase space is spanned by the following field variables on $\mathcal{N}$,

- the spatial components $\boldsymbol{e}_i$ and $\boldsymbol{\omega}_i$ of the dreibein and the spin connection,

- the conical coordinates (T, R, ϕ) defined in some neighbourhood of infinity $\mathcal{U}$.

They are subject to the following boundary conditions,

- in the interior of $\mathcal{N}$, the normal vector $\boldsymbol{n} = \frac{1}{2}\varepsilon^{ij}[\boldsymbol{e}_i, \boldsymbol{e}_j]$ is negative timelike,

- on each particle boundary $\mathcal{B}_\pi$, the tangent component of the dreibein $\boldsymbol{e}_\varphi$ vanishes,

- in the neighbourhood of infinity $\mathcal{U}$, the relation (4.12) is satisfied for some M and S.

From the total Lagrangian, we can read off the symplectic potential and the Hamiltonian. The symplectic potential is the one-form Θ, which is given by those terms of $\mathcal{L}$ which are linear in time derivatives. There is a contribution from the bulk term (4.5), and from the boundary term (4.14) at infinity,

$$\Theta = \frac{1}{8\pi G} \int_{\mathcal{N}} \mathrm{d}^2 x \, \varepsilon^{ij} \, \mathrm{Tr}(\mathrm{d}\boldsymbol{\omega}_i \, \boldsymbol{e}_j) - \frac{M}{2\pi} \int_{\infty} \mathrm{d}s \, \partial_s \phi \, (\mathrm{d}T + 4GS \, \mathrm{d}\phi). \tag{4.16}$$

Here and in the following, the upright letter d always denotes the exterior derivative on the phase space, except when it appears immediately behind an integral, where it denotes the measure on $\mathcal{N}$ or along a curve on $\mathcal{N}$. The Hamiltonian $\mathcal{H}$ is defined by those terms on $\mathcal{L}$ which do not contain time derivatives. There is a contribution from the bulk term, and one contribution from each particle boundary,

$$\mathcal{H} = \sum_{\pi} \zeta_\pi \, \mathcal{C}_\pi - \frac{1}{16\pi G} \int_{\mathcal{N}} \mathrm{d}^2 x \, \varepsilon^{ij} \, \mathrm{Tr}(\boldsymbol{\omega}_t \, \boldsymbol{T}_{ij} + \boldsymbol{e}_t \, \boldsymbol{F}_{ij}). \tag{4.17}$$

This is obviously a linear combination of primary constraints, with the multipliers $\boldsymbol{e}_t$, $\boldsymbol{\omega}_t$, and ζ_π as coefficients. These are not considered as phase space variables, but rather as free parameters entering the Hamiltonian. The *kinematical* constraints are those associated with the multipliers $\boldsymbol{e}_t$ and $\boldsymbol{\omega}_t$,

$$\boldsymbol{F}_{ij} \approx 0, \qquad \boldsymbol{T}_{ij} \approx 0. \tag{4.18}$$

The *dynamical* constraints are the mass shell constraints, associated with the multipliers ζ_π,

$$\mathcal{C}_\pi = \frac{u_\pi - \cos(4\pi G m_\pi)}{16\pi^2 G^2} \approx 0. \tag{4.19}$$

The phase space reduction will now be applied to the kinematical constraints. We are going to solve the constraints (4.18), and divide out the associated gauge symmetries. The mass shell constraints will not be affected by this reduction. It is then already obvious that the reduced Hamiltonian is a linear combination of the mass shell constraints, as all other terms in (4.17) are zero when the kinematical constraints are solved. We may therefore in following only focus on the symplectic structure.

Minkowski coordinates

The general solution to the kinematical constraints (4.18) is well known. Within any simply connected subset of the space manifold, it can be parameterized by a group valued field $\boldsymbol{g} : \mathcal{N} \to \mathsf{SL}(2)$ and a vector field $\boldsymbol{f} : \mathcal{N} \to \mathfrak{sl}(2)$, so that [3, 4]

$$\boldsymbol{\omega}_i = \boldsymbol{g}^{-1} \partial_i \boldsymbol{g}, \qquad \boldsymbol{e}_i = \boldsymbol{g}^{-1} \partial_i \boldsymbol{f} \, \boldsymbol{g}. \tag{4.20}$$

It follows that the metric is given by

$$g_{ij} = \tfrac{1}{2} \mathrm{Tr}(\boldsymbol{e}_i \boldsymbol{e}_j) = \tfrac{1}{2} \mathrm{Tr}(\partial_i \boldsymbol{f} \partial_j \boldsymbol{f}), \tag{4.21}$$

and the normal vector becomes

$$\boldsymbol{n} = \tfrac{1}{2} \varepsilon^{ij} \, [\boldsymbol{e}_i, \boldsymbol{e}_j] = \tfrac{1}{2} \varepsilon^{ij} \, \boldsymbol{g}^{-1} [\partial_i \boldsymbol{f}, \partial_j \boldsymbol{f}] \, \boldsymbol{g}. \tag{4.22}$$

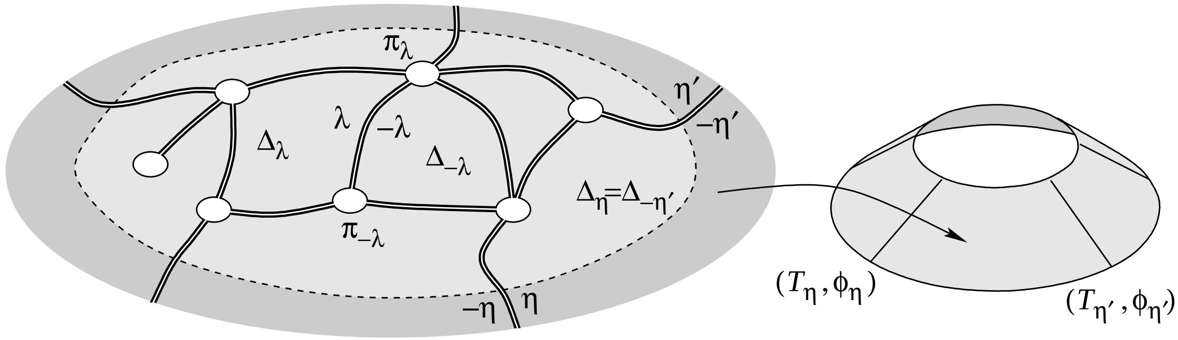

Figure 10: The triangulated space manifold $\mathcal{N}$ is covered by a collection of simply connected polygons $\Delta \in \mathit{\Pi}$. A neighbourhood of infinity $\mathcal{U}$ is divided into simply connected segments $\mathcal{U} \cap \Delta$, one for each non-compact polygon $\Delta \in \mathit{\Pi}_\infty$. The links are assumed to be spacetime geodesics, and in $\mathcal{U}$ the external links $\eta \in \mathit{\Gamma}_\infty$ are spatial half lines in the spinning cone, which conical coordinates $T \to T_\eta$ and $\phi \to \phi_\eta$.

The field $\boldsymbol{f}$ defines an *isometric embedding* of the space manifold into Minkowski space. This is just a different way to define the Minkowski charts that we already introduced in section 2. The components of $\boldsymbol{f}$ are the Minkowski coordinates. The field $\boldsymbol{g}$ represents a Lorentz rotation at each point in space. It tells us how the local frame defined by the dreibein at a point in space is related to the Minkowski frame defined in the respective chart.

If we want to parameterize the solutions to the kinematical constraints globally on $\mathcal{N}$, then we have to cover the space manifold by a collection of simply connected subsets. We therefore introduce a *triangulation*, in the very same way as in section 1. The only difference is that the links $\lambda \in \mathit{\Gamma}$ are now oriented curves on $\mathcal{N}$, and the vertices are the particle boundaries $\mathcal{B}_\pi$. The internal links $\lambda \in \mathit{\Gamma}_0$ are curves beginning at a particle boundary $\mathcal{B}_{\pi_{-\lambda}}$, and ending at a particle boundary $\mathcal{B}_{\pi_\lambda}$. The external links $\eta \in \mathit{\Gamma}_\infty$ extend from a particle boundary $\mathcal{B}_{\pi_{-\eta}}$ to infinity. The polygons $\Delta \in \mathit{\Pi}$ are simply connected, closed subsets of $\mathcal{N}$, overlapping along the links. A typical such triangulation is shown in figure 10.

Now, suppose we are given a field configuration $\boldsymbol{e}_i$ and $\boldsymbol{\omega}_i$ on $\mathcal{N}$, satisfying the kinematical constraints $\boldsymbol{F}_{ij} = 0$ and $\boldsymbol{T}_{ij} = 0$, and a graph $\mathit{\Gamma}$ defining a triangulation of $\mathcal{N}$. We can then introduce a pair of fields $\boldsymbol{g}_\Delta : \Delta \to \mathsf{SL}(2)$ and $\boldsymbol{f}_\Delta : \Delta \to \mathfrak{sl}(2)$ in each polygon, so that

$$\boldsymbol{\omega}_i|_\Delta = \boldsymbol{g}_\Delta^{-1} \partial_i \boldsymbol{g}_\Delta, \qquad \boldsymbol{e}_i|_\Delta = \boldsymbol{g}_\Delta^{-1} \partial_i \boldsymbol{f}_\Delta \, \boldsymbol{g}_\Delta. \tag{4.23}$$

The image $\boldsymbol{f}_\Delta(\Delta)$ becomes a spacelike surface embedded into Minkowski space. We call this the *embedded polygon*. The embedded polygons are those in figure 3, provided that we impose some further boundary condition on the fields $\boldsymbol{g}_\Delta$ and $\boldsymbol{f}_\Delta$.

The first observation is that the fields $\boldsymbol{g}_\Delta$ and $\boldsymbol{f}_\Delta$ are not uniquely determined by the dreibein and the spin connection. The right hand sides of (4.23) are invariant under the transformations

$$\boldsymbol{g}_\Delta \mapsto \boldsymbol{h}_\Delta \, \boldsymbol{g}_\Delta, \qquad \boldsymbol{f}_\Delta \mapsto \boldsymbol{h}_\Delta \, (\boldsymbol{f}_\Delta - \boldsymbol{v}_\Delta) \, \boldsymbol{h}_\Delta^{-1}, \tag{4.24}$$

where $\boldsymbol{h}_\Delta \in \mathsf{SL}(2)$ and $\boldsymbol{v}_\Delta \in \mathfrak{sl}(2)$ are two independent *constants* for each polygon $\Delta \in \mathit{\Pi}$. This is a Poincaré transformation, thus a coordinate transformation in the respective coordinate charts. On the embedded polygon $\boldsymbol{f}_\Delta(\Delta)$, it acts as a Lorentz rotation by $\boldsymbol{h}_\Delta$, and a translation by $\boldsymbol{v}_\Delta$. Clearly, these are the redundancy transformations already considered in section 2.

Link variables

Referring to the embedding fields $\boldsymbol{g}_\Delta$ and $\boldsymbol{f}_\Delta$, we can now also define the link variables. When considered as a subset of $\mathcal{N}$, a polygon Δ is not only bounded by the edges $\lambda \in \mathit{\Gamma}_\Delta$, but also by some segments of the particles boundaries $\Delta \cap \mathcal{B}_\pi$. The point particle condition (4.3) implies that the field $\boldsymbol{f}_\Delta$ is constant on these segments,

$$\boldsymbol{e}_\varphi|_{\Delta\cap\mathcal{B}_\pi} = \boldsymbol{g}_\Delta^{-1} \partial_\varphi \boldsymbol{f}_\Delta \, \boldsymbol{g}_\Delta|_{\mathcal{B}_\pi} = 0 \quad \Rightarrow \quad \boldsymbol{f}_\Delta|_{\mathcal{B}_\pi} = \boldsymbol{x}_{\pi,\Delta}. \tag{4.25}$$

The constant $\boldsymbol{x}_{\pi,\Delta}$ is called the *position* of the particle π in the polygon Δ. There are as many positions $\boldsymbol{x}_{\pi,\Delta}$ of the particle π as there are polygons Δ adjacent to π, and these position are the corners of the embedded polygons in figure 3. We also see that the point particle condition indeed implies that the particles are pointlike.

Now, let us assume that all the links are geodesics in spacetime. This can be regarded as a gauge condition in general relativity, restricting the possible foliations and also the embeddings of the triangulated space manifold $\mathcal{N}$ into the spacetime. If this is the case, then the image $\boldsymbol{f}_{\Delta_\eta}(\eta)$ of each link is a straight line in Minkowski space. Let us introduce on λ an affine parameter s. For internal links $\lambda \in \mathit{\Gamma}_0$, it is chosen so that $s \in [0,1]$, and $\lambda(0) = \pi_{-\lambda}$ and $\lambda(1) = \pi_\lambda$. For external links, we choose $s \in [0, infty)$, and for reversed external links $s \in (-\infty, 0]$, and in these cases s measures the proper length. Then we define

$$\boldsymbol{z}_\eta = \partial_s \boldsymbol{f}_{\Delta_\eta}, \qquad \eta \in \mathit{\Gamma}. \tag{4.26}$$

The vector $\boldsymbol{z}_\lambda$ is a constant, because the link λ is a geodesic. For external links, it is a spacelike unit vector. For internal links, it is the relative position vector,

$$\boldsymbol{z}_\lambda = \boldsymbol{x}_{\pi_\lambda,\Delta_\lambda} - \boldsymbol{x}_{\pi_{-\lambda},\Delta_\lambda}, \qquad \lambda \in \mathit{\Gamma}_0. \tag{4.27}$$

The transition functions are defined as follows. Consider a pair of links $\pm\eta$, and the adjacent polygons $\Delta_{\pm\eta}$. The both links represent the same curve on $\mathcal{N}$, and therefore the dreibein and the spin connection are equal on λ and $-\lambda$. But the fields $\boldsymbol{g}_{\Delta_\lambda}$ and $\boldsymbol{f}_{\Delta_\lambda}$ on λ, and $\boldsymbol{g}_{\Delta_{-\lambda}}$ and $\boldsymbol{f}_{\Delta_{-\lambda}}$ on $-\lambda$ are in general different. On the other hand, we have seen above that if two embeddings define the same dreibein and the same spin connection, then they are related by a Poincaré transformation of the form (4.24). Hence, for each pair of links $\pm\eta$ there exists a constant $\boldsymbol{g}_\lambda \in \mathsf{SL}(2)$ and a constant $\boldsymbol{f}_\lambda \in \mathfrak{sl}(2)$ so that

$$\boldsymbol{g}_{\Delta_{-\lambda}} = \boldsymbol{g}_\lambda \, \boldsymbol{g}_{\Delta_\lambda}, \qquad \boldsymbol{f}_{\Delta_{-\lambda}} = \boldsymbol{g}_\lambda \, (\boldsymbol{f}_{\Delta_\lambda} - \boldsymbol{f}_\lambda) \, \boldsymbol{g}_\lambda^{-1}, \qquad \lambda \in \mathit{\Gamma}. \tag{4.28}$$

Exchanging Δ_λ and $\Delta_{-\lambda}$ in the first equation, and differentiating the second equation with respect to the curve parameter s, where we have to take into account that the reversed link is oriented into the opposite direction, we derive the following relations,

$$\boldsymbol{g}_{-\lambda} = \boldsymbol{g}_\lambda^{-1}, \qquad \boldsymbol{z}_{-\lambda} = -\boldsymbol{g}_\lambda \, \boldsymbol{z}_\lambda \, \boldsymbol{g}_\lambda^{-1}, \qquad \lambda \in \mathit{\Gamma}. \tag{4.29}$$

Hence $\boldsymbol{g}_\lambda$ is the transition function, and the relation (3.3) is recovered for both internal and external links. The translational component $\boldsymbol{f}_\lambda$ of the transition function has never shown up in section 2, because there are link variables that refer to the absolute positions of the polygon in the embedding Minkowski space.

The conical frame

There is one more gauge condition that we have to impose. The external links must be spatial half lines in the spinning cone. Hence, if we follow a curve $\eta \in \mathit{\Gamma}_\infty$ on $\mathcal{N}$ to infinity, then at some point the curves enters the neighbourhood of infinity $\mathcal{U}$, and in the limit the conical coordinates are given by

$$T \to T_\eta, \qquad \phi \to \phi_\eta, \qquad R \to \infty, \qquad \eta \in \mathit{\Gamma}_\infty. \tag{4.30}$$

Moreover, there is also a further condition to be imposed on the embedding of the non-compact polygons. Let us choose the neighbourhood of infinity $\mathcal{U}$ so that, for each non-compact polygon $\Delta \in \mathit{\Pi}_\infty$, the intersection $\mathcal{U} \cap \Delta$ is a simply connected segment, as indicated in figure 10. This is always possible, because given any neighbourhood of infinity we can always go over to a smaller one which has this property.

The segment $\mathcal{U} \cap \Delta$ is then not only embedded into the spinning cone, but also into Minkowski space. The first embedding is provided by the conical coordinates (T, R, ϕ), the second by the field $\boldsymbol{f}_\Delta$. And we still have the freedom to apply a Poincaré transformation to $\boldsymbol{f}_\Delta$. Using this, we can always achieve that the two embeddings are related by a local isometry of the form (2.19). There is then the following relation between the conical coordinates (T, R, ϕ), and the Minkowski coordinates $\boldsymbol{f}_\Delta$ in $\mathcal{U} \cap \Delta$,

$$\boldsymbol{f}_\Delta = (T + \tau_\Delta(\phi))\, \boldsymbol{\gamma}_0 + R\, \boldsymbol{\gamma}(\phi - \alpha_\Delta(\phi)), \tag{4.31}$$

where τ_Δ and α_Δ are two linear functions satisfying

$$\tau'_\Delta(\phi) = 4GS, \qquad \alpha'_\Delta(\phi) = 4GM. \tag{4.32}$$

For the fall off condition (4.12) to be satisfied, the field $\boldsymbol{g}_\Delta$ is then necessarily given by

$$\boldsymbol{g}_\Delta = \mathrm{e}^{-\alpha_\Delta(\phi)\boldsymbol{\gamma}_0/2}. \tag{4.33}$$

Using this, we can finally also derive the special transition functions $\boldsymbol{g}_{\pm\eta}$, and the unit vectors $\boldsymbol{z}_{\pm\eta}$ for the external links $\eta \in \mathit{\Gamma}_\infty$. For this purpose, we have to evaluate the fields $\boldsymbol{g}_\Delta$ and $\boldsymbol{f}_\Delta$ on the external links $\pm\eta$, in the limit $R \to \infty$. What we find is

$$\begin{aligned} \boldsymbol{g}_{\Delta_{-\eta}} &\to \mathrm{e}^{\theta^-_\eta \boldsymbol{\gamma}_0/2}, \qquad & \boldsymbol{f}_{\Delta_{-\eta}} &\to (T_\eta + \tau_{\Delta_{-\eta}}(\phi_\eta))\, \boldsymbol{\gamma}_0 + R\, \boldsymbol{\gamma}(\phi_\eta + \theta^-_\eta), \\ \boldsymbol{g}_{\Delta_\eta} &\to \mathrm{e}^{\theta^+_\eta \boldsymbol{\gamma}_0/2}, \qquad & \boldsymbol{f}_{\Delta_\eta} &\to (T_\eta + \tau_{\Delta_\eta}(\phi_\eta))\, \boldsymbol{\gamma}_0 + R\, \boldsymbol{\gamma}(\phi_\eta + \theta^+_\eta), \end{aligned} \tag{4.34}$$

where we introduced the *deviations* already defined in (2.24), thus

$$\theta^+_\eta = -\alpha_{\Delta_\eta}(\phi_\eta), \qquad \theta^-_\eta = -\alpha_{\Delta_{-\eta}}(\phi_\eta). \tag{4.35}$$

Now, consider first the vectors $\boldsymbol{z}_{\pm\eta}$, as defined in (4.27). In the limit $R \to 0$, the conical coordinate R asymptotically defines the proper length on a spatial half line, and therefore we can use $s = \pm R$ as a parameter on the links $\pm\eta$. It then follows from (4.34) that

$$\boldsymbol{z}_\eta = \boldsymbol{\gamma}(\phi_\eta + \theta^+_\eta), \qquad \boldsymbol{z}_{-\eta} = -\boldsymbol{\gamma}(\phi_\eta + \theta^-_\eta), \qquad \eta \in \Gamma_\infty, \tag{4.36}$$

which is the same as (3.5). To find the transition functions $\boldsymbol{g}_{\pm\eta}$, we have to use the defining relation (4.28), which can also be evaluated in the limit $R \to 0$. What we get is

$$\boldsymbol{g}_\eta = \boldsymbol{g}_{\Delta_{-\eta}} \, \boldsymbol{g}^{-1}_{\Delta_\eta} = \mathrm{e}^{-(\theta^+_\eta - \theta^-_\eta)\boldsymbol{\gamma}_0/2}. \tag{4.37}$$

We can therefore make the definition

$$8\pi G M_\eta = \theta^+_\eta - \theta^-_\eta \quad \Rightarrow \quad \boldsymbol{g}_\eta = \mathrm{e}^{-4\pi G M_\eta \boldsymbol{\gamma}_0}, \tag{4.38}$$

and recover the definition (3.4) of the energy M_η, and also the second equation in (3.6). The first equation in (3.6) follows directly from the definition (4.35). With $\Delta = \Delta_\eta = \Delta_{-\eta'}$ we have

$$\theta^-_{\eta'} - \theta^+_\eta = -\alpha_\Delta(\phi_{\eta'}) + \alpha_\Delta(\phi_\eta) = -4GM\,(\phi_{\eta'} - \phi_\eta). \tag{4.39}$$

Finally, what remains is the average condition (3.7). In section 2 we have seen that this is equivalent to the condition that the integral of the functions $\alpha(\phi)$ over the circle at infinity vanishes, hence

$$\int_\infty \mathrm{d}s\, \partial_s\phi\, \alpha(\phi) = \sum_{\Delta\in\Pi_\infty} \int_{\infty\cap\Delta} \mathrm{d}s\, \partial_s\phi\, \alpha_\Delta(\phi) = 0. \tag{4.40}$$

This is the same as (2.28), we only changed the notation slightly, because now we have consider this as an integral along a curve on $\mathcal{N}$, and ϕ is a field variable on $\mathcal{U} \subset \mathcal{N}$. The notation $\infty \cap \Delta$ means that the integral is evaluated along the segment of the circle at infinity which belongs to the non-compact polygon Δ.

The symplectic structure

With these formulas at hand, we can now compute the symplectic structure on the subspace defined by the kinematical constraints (4.18). The embedding fields $\boldsymbol{g}_\Delta$ and $\boldsymbol{f}_\Delta$ can be used as coordinates on this subspace. To derive the symplectic potential, we insert the parameterization (4.23) into the original expression (4.16),

$$\Theta = \frac{1}{8\pi G} \int_{\mathcal{N}} \mathrm{d}^2x\, \varepsilon^{ij}\, \mathrm{Tr}(\mathrm{d}\boldsymbol{\omega}_i\, \boldsymbol{e}_j) - \frac{M}{2\pi} \int_\infty \mathrm{d}s\, \partial_s\phi\, (\mathrm{d}T + 4GS\, \mathrm{d}\phi). \tag{4.41}$$

For simplicity, let us first ignore all boundary terms at infinity, or consider a closed universe where these are absent. The symplectic potential is then given by the bulk term only. To write this as a function of the embedding fields $\boldsymbol{g}_\Delta$ and $\boldsymbol{f}_\Delta$, we first have to split the integral into a sum of integrals over the individual polygons,

$$\Theta_0 = \frac{1}{8\pi G} \sum_{\Delta\in\Pi} \int_\Delta \mathrm{d}^2x\, \varepsilon^{ij}\, \mathrm{Tr}(\mathrm{d}\boldsymbol{\omega}_i\, \boldsymbol{e}_j) = \frac{1}{8\pi G} \sum_{\Delta\in\Pi} \int_\Delta \mathrm{d}^2x\, \varepsilon^{ij}\, \partial_i \mathrm{Tr}(\mathrm{d}\boldsymbol{g}_\Delta\, \boldsymbol{g}^{-1}_\Delta\, \partial_j \boldsymbol{f}_\Delta). \tag{4.42}$$

Here we have inserted (4.23), and performed some simple algebraic manipulations. The integrand is thus a total derivative, and we can write the result as a boundary integral,

$$\Theta_0 = \frac{1}{8\pi G} \sum_{\Delta \in \Pi} \int_{\partial\Delta} \mathrm{d}s \, \mathrm{Tr}(\mathrm{d}\boldsymbol{g}_\Delta \, \boldsymbol{g}_\Delta^{-1} \partial_s \boldsymbol{f}_\Delta). \tag{4.43}$$

The boundary $\partial\Delta$ of the polygon Δ is always traversed in counter clockwise direction. It consists of the edges $\lambda \in \Gamma_\Delta$, and the segments of the particles boundaries belonging to the polygon Δ. The particle boundaries, however, do not contribute to the integral, because according to the point particle condition $\boldsymbol{f}_\Delta$ is constant, and therefore the integrand vanishes. What remains is

$$\Theta_0 = \frac{1}{8\pi G} \sum_{\Delta \in \Pi} \sum_{\lambda \in \Gamma_\Delta} \int_\lambda \mathrm{d}s \, \mathrm{Tr}(\mathrm{d}\boldsymbol{g}_\Delta \, \boldsymbol{g}_\Delta^{-1} \partial_s \boldsymbol{f}_\Delta). \tag{4.44}$$

Now, we may equally well sum over all links $\lambda \in \Gamma$, because each link appears exactly once as a boundary of a polygon,

$$\Theta_0 = \frac{1}{8\pi G} \sum_{\lambda \in \Gamma} \int_\lambda \mathrm{d}s \, \mathrm{Tr}(\mathrm{d}\boldsymbol{g}_{\Delta_\lambda} \, \boldsymbol{g}_{\Delta_\lambda}^{-1} \partial_s \boldsymbol{f}_{\Delta_\lambda}). \tag{4.45}$$

For a closed universe, all links are internal links, thus $\Gamma = \Gamma_0$. Using this and the decomposition $\Gamma_0 = \Gamma_+ \cap \Gamma_-$, we can also write

$$\Theta_0 = \frac{1}{8\pi G} \sum_{\lambda \in \Gamma_+} \int_\lambda \mathrm{d}s \, \mathrm{Tr}(\mathrm{d}\boldsymbol{g}_{\Delta_\lambda} \, \boldsymbol{g}_{\Delta_\lambda}^{-1} \, \partial_s \boldsymbol{f}_{\Delta_\lambda} - \mathrm{d}\boldsymbol{g}_{\Delta_{-\lambda}} \, \boldsymbol{g}_{\Delta_{-\lambda}}^{-1} \, \partial_s \boldsymbol{f}_{\Delta_{-\lambda}}). \tag{4.46}$$

We can then use the defining relations (4.28) of the transition functions, to express $\boldsymbol{g}_{\Delta_{-\lambda}}$ and $\boldsymbol{f}_{\Delta_{-\lambda}}$ in terms of $\boldsymbol{g}_{\Delta_\lambda}$ and $\boldsymbol{f}_{\Delta_\lambda}$. This gives

$$\Theta_0 = -\frac{1}{8\pi G} \sum_{\lambda \in \Gamma_+} \int_\lambda \mathrm{d}s \, \mathrm{Tr}(\boldsymbol{g}_\lambda^{-1} \, \mathrm{d}\boldsymbol{g}_\lambda \, \partial_s \boldsymbol{f}_{\Delta_\lambda}). \tag{4.47}$$

Now, the transition function $\boldsymbol{g}_\lambda$ is a constant, and therefore the integrand is again a total derivative. Using this and the definition (4.27) of the relative position vector $\boldsymbol{z}_\lambda$, we get

$$\Theta_0 = -\frac{1}{8\pi G} \sum_{\lambda \in \Gamma_+} \mathrm{Tr}(\boldsymbol{g}_\lambda^{-1} \, \mathrm{d}\boldsymbol{g}_\lambda \, \boldsymbol{z}_\lambda). \tag{4.48}$$

So, for a closed universe we have shown that the symplectic potential, derived from the Einstein Hilbert action, is indeed given by the expression (3.9). The terms associated with the external links are in this case not present.

For an open universe, we perform the same calculation, but we also have to take into account some boundary terms at infinity, and the special properties of the external links. Let us first consider the external links, still ignoring the boundary terms at infinity. The total symplectic potential can then still be written as (4.45), but in (4.46) and consequently also in (4.47) we have to sum over all external links as well. We shall now show, however, that the external links do not

contribute to this sum. Consider an external link $\eta \in \mathit{\Gamma}_\infty$, and the corresponding term in (4.47), which is given by

$$-\frac{1}{8\pi G}\int_\eta \mathrm{d}s\,\mathrm{Tr}(\boldsymbol{g}_\eta^{-1}\,\mathrm{d}\boldsymbol{g}_\eta\,\partial_s\boldsymbol{f}_{\Delta_\eta}). \tag{4.49}$$

The integrand is still a total derivative, but the integral cannot be expressed like (4.48), because a relative position vector assigned to the external link η does not exist. However, the integrand is actually zero. This can be seen as follows. According to (4.37), the transition function $\boldsymbol{g}_\eta$ is an element of the subgroup $\mathsf{SO}(2)$, which implies that the term $\boldsymbol{g}_\eta^{-1}\,\mathrm{d}\boldsymbol{g}_\eta$ is proportional to $\boldsymbol{\gamma}_0$. On the other hand, according to (4.26) and (4.36), the derivative $\partial_s\boldsymbol{f}_{\Delta_\eta}$ is orthogonal to $\boldsymbol{\gamma}_0$, and thus the integrand is zero.

So, there is no contribution to the symplectic potential from the external links. The only additional terms for an open universe are the boundary terms at spatial infinity. There are two types of such boundary terms. Let us first consider the boundary term in (4.41). It is an integral along the circle at infinity. We split it into segments belonging to the external polygons $\Delta \in \mathit{\Pi}_\infty$,

$$-\frac{1}{8\pi G}\sum_{\Delta\in\mathit{\Pi}_\infty}\int_{\infty\cap\Delta}\mathrm{d}s\,4GM\,\partial_s\phi\,(\mathrm{d}T + 4GS\,\mathrm{d}\phi). \tag{4.50}$$

Remember that the abbreviated notation $\infty \cap \Delta$ means that the integral in evaluated along the circle segment which defines the boundary at infinity of the non-compact polygon $\Delta \in \mathit{\Pi}_\infty$. The integrand is written in this particular form, because according to (4.32) we have

$$4GM\,\partial_s\phi = \partial_s\alpha_\Delta(\phi). \tag{4.51}$$

Hence, (4.50) is equal to

$$-\frac{1}{8\pi G}\sum_{\Delta\in\mathit{\Pi}_\infty}\int_{\infty\cap\Delta}\mathrm{d}s\,\partial_s\alpha_\Delta(\phi)\,(\mathrm{d}T + 4GS\,\mathrm{d}\phi). \tag{4.52}$$

There is then a second boundary term at spatial infinity, which arises from the boundary integral (4.43). The boundary of each non-compact polygon also includes a segment of the circle at infinity, which we did not take into account so far. Hence, yet another contribution to the symplectic potential is

$$\frac{1}{8\pi G}\sum_{\Delta\in\mathit{\Pi}_\infty}\int_{\infty\cap\Delta}\mathrm{d}s\,\mathrm{Tr}(\mathrm{d}\boldsymbol{g}_\Delta\,\boldsymbol{g}_\Delta^{-1}\partial_s\boldsymbol{f}_\Delta). \tag{4.53}$$

The path along which this integral is evaluated lies entirely within the segment $\mathcal{U}\cap\Delta$ of the neighbourhood of infinity. We can therefore insert the expressions (4.31) for $\boldsymbol{f}_\Delta$ and (4.33) for $\boldsymbol{g}_\Delta$. Using this, one finds that (4.53) is equal to

$$\frac{1}{8\pi G}\sum_{\Delta\in\mathit{\Pi}_\infty}\int_{\infty\cap\Delta}\mathrm{d}s\;\mathrm{d}\left(\alpha(\phi)\right)\,\left(\partial_s T + 4GS\,\partial_s\phi\right). \tag{4.54}$$

Let us add to this a total exterior derivative, which we are always allowed to, as the only relevant object is the symplectic two-form $\mathit{\Omega} = \mathrm{d}\mathit{\Theta}$. Then we have

$$-\frac{1}{8\pi G}\sum_{\Delta\in\mathit{\Pi}_\infty}\int_{\infty\cap\Delta}\mathrm{d}s\,\alpha_\Delta(\phi)\,[\partial_s(\mathrm{d}T + 4GS\,\mathrm{d}\phi) + 4G\,\mathrm{d}S\,\partial_s\phi]. \tag{4.55}$$

The term proportional to $\mathrm{d}S$ is the expression which is set to zero in (4.40). It therefore vanishes. What remains is, together with (4.52), again a total derivative,

$$\Theta_\infty = -\frac{1}{8\pi G} \sum_{\Delta \in \Pi_\infty} \int_{\infty \cap \Delta} \mathrm{d}s \, \partial_s [\alpha_\Delta(\phi) \, (\mathrm{d}T + 4GS \, \mathrm{d}\phi)]. \tag{4.56}$$

The integral can be evaluated and becomes

$$-\frac{1}{8\pi G} \sum_{\Delta \in \Pi_\infty} [\alpha_\Delta(\phi_{\eta'}) \, (\mathrm{d}T_{\eta'} + 4GS \, \mathrm{d}\phi_{\eta'}) - \alpha_\Delta(\phi_\eta) \, (\mathrm{d}T_\eta + 4GS \, \mathrm{d}\phi_\eta)]. \tag{4.57}$$

Here we used that the segment of the circle at infinity belonging to the polygon Δ is bounded by the two external link $\eta \in \Gamma_\infty$ and $-\eta \in \Gamma_{-\infty}$, where $\Delta = \Delta_\eta = \Delta_{-\eta'}$. And we inserted the conical coordinates on these edges, which are given by (4.30). Finally, we can use the definitions (4.35) and rearrange the summation, which gives

$$\Theta_\infty = -\frac{1}{8\pi G} \sum_{\eta \in \Gamma_\infty} (\theta_\eta^+ - \theta_\eta^-) \, (\mathrm{d}T_\eta + 4GS \, \mathrm{d}\phi_\eta). \tag{4.58}$$

And now, we just have to use (4.38) and make another definition, namely

$$L_\eta = -4GS \, M_\eta, \tag{4.59}$$

to obtain the final result. The contribution to the symplectic potential from the boundary at infinity is, again up to a total derivative,

$$\Theta_\infty = -\sum_{\eta \in \Gamma_\infty} M_\eta \, (\mathrm{d}T_\eta + 4GS \, \mathrm{d}\phi_\eta) = \sum_{\eta \in \Gamma_\infty} (T_\eta \, \mathrm{d}M_\eta + L_\eta \, \mathrm{d}\phi_\eta). \tag{4.60}$$

This completes the technical part of the derivation. The sum $\Theta = \Theta_\infty + \Theta_0$ is the same as (3.9). Now we just have to sort out what this equality actually means, and how it fits together with the definitions made in section 3. First of all, it follows that all those variations of the embedding fields $\boldsymbol{g}_\Delta$ and $\boldsymbol{f}_\Delta$ are gauge symmetries of general relativity, which preserve the values of link variables showing up in (4.48) and (4.60). In other words, the link variables are the *observables* of the particle model, in the sense that all physical degrees of freedom are encoded in these variables. This is what we had to assume in the beginning of section 3.

Let us consider the quotient space of all field configurations $\boldsymbol{g}_\Delta$ and $\boldsymbol{f}_\Delta$, divided by the gauge transformations that preserve the values of the link variables. This quotient space is obviously the kinematical phase space $\mathcal{S}_\Gamma$ defined in section 3. To see this, we first note that by definition the observables, hence the link variables are the coordinates on this space. And furthermore, they are not independent but subject to the kinematical constraints (3.14-3.16). It is therefore not the extended phase space $\mathcal{Q}_\Gamma$ which is recovered here, but the kinematical subspace $\mathcal{S}_\Gamma \subset \mathcal{Q}_\Gamma$. To check this, we verify that the constraints defined in section 3 are indeed satisfied by the link variables introduced above.

The constraint $\boldsymbol{Z}_\Delta \approx 0$ for each compact polygon $\Delta \in \Pi_0$ is an immediate consequence of the definition (4.27). The same applies to the constraint $\mathcal{Z}_\Delta \approx 0$ for each non-compact polygon $\Delta \in \Pi_\infty$. This follows from the definition of the clocks in (4.30), the fall off condition (4.31) for $\boldsymbol{f}_\Delta$ at infinity, and again the definition (4.27) of the vectors $\boldsymbol{z}_\lambda$. And finally, the constraint $\mathcal{J}_\eta \approx 0$ for each external link $\eta \in \Gamma_\infty$ simply follows from the definition (4.59) of the auxiliary variable L_η. So, everything fits together with the definitions made in section 3.

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Appendix

Here we summarize some facts about the spinor representation of the three dimensional Lorentz algebra $\mathfrak{sl}(2)$ of traceless 2×2 matrices, and the associated Lie group $\mathsf{SL}(2)$. A more comprehensive overview, using the same notion, is given in [13]. As a vector space, $\mathfrak{sl}(2)$ is isometric to three dimensional Minkowski space. An orthonormal basis is given by the gamma matrices

$$\boldsymbol{\gamma}_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \boldsymbol{\gamma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \boldsymbol{\gamma}_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.1}$$

They satisfy

$$\boldsymbol{\gamma}_a \boldsymbol{\gamma}_b = \eta_{ab} \, \mathbf{1} - \varepsilon_{abc} \, \boldsymbol{\gamma}^c, \tag{A.2}$$

where $a, b = 0, 1, 2$. The metric η_{ab} has signature $(-,+,+)$, and for the Levi Civita symbol ε^{abc} we have $\varepsilon^{012} = 1$. Expanding a matrix in terms of the gamma matrices, we obtain an isomorphism of $\mathfrak{sl}(2)$ and Minkowski space,

$$\boldsymbol{v} = v^a \, \boldsymbol{\gamma}_a \qquad \Leftrightarrow \qquad v^a = \tfrac{1}{2} \mathrm{Tr}(\boldsymbol{v} \boldsymbol{\gamma}^a). \tag{A.3}$$

Some useful relations are that the scalar product of two vectors is equal to the trace norm of the corresponding matrices, and the vector product is essentially given by the matrix commutator,

$$\tfrac{1}{2} \mathrm{Tr}(\boldsymbol{v} \boldsymbol{w}) = v_a w^a, \qquad \tfrac{1}{2} [\boldsymbol{v}, \boldsymbol{w}] = -\varepsilon^{abc} \, v_a w_b \, \boldsymbol{\gamma}_c. \tag{A.4}$$

Sometimes it is useful to introduce cylindrical coordinates in Minkowski space, writing

$$\boldsymbol{v} = \tau \, \boldsymbol{\gamma}_0 + \rho \, \boldsymbol{\gamma}(\varphi), \tag{A.5}$$

where τ is the time, $\rho \geq 0$ the radial, and φ the angular coordinate, which is redundant for $\rho = 0$ and has a period of 2π. The unit vector $\boldsymbol{\gamma}(\varphi)$ defines the angular direction φ in Minkowski space. This vector and its derivative $\boldsymbol{\gamma}'(\varphi)$ are two orthogonal unit spacelike vectors, forming together with $\boldsymbol{\gamma}_0$ a rotated orthonormal basis,

$$\boldsymbol{\gamma}(\varphi) = \cos\varphi \, \boldsymbol{\gamma}_1 + \sin\varphi \, \boldsymbol{\gamma}_2, \qquad \boldsymbol{\gamma}'(\varphi) = \cos\varphi \, \boldsymbol{\gamma}_2 - \sin\varphi \, \boldsymbol{\gamma}_1. \tag{A.6}$$

Useful relations are

$$\boldsymbol{\gamma}_0 \boldsymbol{\gamma}(\varphi) = \boldsymbol{\gamma}'(\varphi), \qquad \boldsymbol{\gamma}_0 \boldsymbol{\gamma}'(\varphi) = -\boldsymbol{\gamma}(\varphi), \tag{A.7}$$

and

$$\boldsymbol{\gamma}(\varphi_1) \, \boldsymbol{\gamma}(\varphi_2) = \cos(\varphi_1 - \varphi_2) \, \mathbf{1} + \sin(\varphi_1 - \varphi_2) \, \boldsymbol{\gamma}_0. \tag{A.8}$$

The Lie group $\mathsf{SL}(2)$ consists of matrices $\boldsymbol{u}$ with unit determinant. The group acts on the algebra in the adjoint representation, so that

$$\boldsymbol{v} \mapsto \boldsymbol{u}^{-1}\boldsymbol{v}\boldsymbol{u}. \tag{A.9}$$

This defines a proper Lorentz rotation of the vector $\boldsymbol{v}$. The conjugacy classes of the algebra are characterized by the invariant length $v^a v_a = \frac{1}{2}\mathrm{Tr}(\boldsymbol{v}^2)$. For timelike and lightlike vectors $\boldsymbol{v}$ (with $\frac{1}{2}\mathrm{Tr}(\boldsymbol{v}^2) \leq 0$) we distinguish between positive ($v^0 = \frac{1}{2}\mathrm{Tr}(\boldsymbol{v}\boldsymbol{\gamma}^0) > 0$) and negative ($v^0 = \frac{1}{2}\mathrm{Tr}(\boldsymbol{v}\boldsymbol{\gamma}^0) < 0$) timelike and lightlike vectors, respectively. Special group elements are those representing spatial rotations about the $\boldsymbol{\gamma}_0$-axis. The matrix

$$\boldsymbol{u} = \mathrm{e}^{-\omega\boldsymbol{\gamma}_0/2} = \cos(\omega/2)\,\mathbf{1} - \sin(\omega/2)\,\boldsymbol{\gamma}_0, \tag{A.10}$$

represents a rotation by ω in counter clockwise direction,

$$\boldsymbol{\gamma}_0 \mapsto \boldsymbol{u}^{-1}\boldsymbol{\gamma}_0\,\boldsymbol{u} = \boldsymbol{\gamma}_0, \qquad \boldsymbol{\gamma}(\varphi) \mapsto \boldsymbol{u}^{-1}\boldsymbol{\gamma}(\varphi)\,\boldsymbol{u} = \boldsymbol{\gamma}(\varphi+\omega). \tag{A.11}$$

A generic element $\boldsymbol{u} \in \mathsf{SL}(2)$ can be expanded in terms of the unit and the gamma matrices, defining a scalar u and a vector $\boldsymbol{p} \in \mathfrak{sl}(2)$,

$$\boldsymbol{u} = u\,\mathbf{1} + p^a\,\boldsymbol{\gamma}_a \quad \Rightarrow \quad \boldsymbol{p} = p^a\,\boldsymbol{\gamma}_a. \tag{A.12}$$

The scalar u is half of the trace of $\boldsymbol{u}$, and the vector $\boldsymbol{p}$ is called the *projection* of $\boldsymbol{u}$. The determinant condition implies that

$$u^2 = p^a p_a + 1. \tag{A.13}$$

According to the property of the vector $\boldsymbol{p}$, we distinguish between timelike, lightlike and spacelike group elements $\boldsymbol{u}$. For a timelike element we have $-1 < u < 1$, and it represents a rotation about some timelike axis, which is specified by the vector $\boldsymbol{p}$. The angle of rotation ω obeys $u = \cos(\omega/2)$. Lightlike elements are null rotations, and spacelike elements are boosts.

Finally, a parity transformation, which acts on Minkowski space as a reflection of space, can be defined as follows,

$$\boldsymbol{v} \mapsto -\boldsymbol{\gamma}_2\,\boldsymbol{v}\,\boldsymbol{\gamma}_2. \tag{A.14}$$

It represents a reflection of the $\boldsymbol{\gamma}_2$-axis,

$$\boldsymbol{\gamma}_0 \mapsto \boldsymbol{\gamma}_0, \qquad \boldsymbol{\gamma}_1 \mapsto \boldsymbol{\gamma}_1, \qquad \boldsymbol{\gamma}_2 \mapsto -\boldsymbol{\gamma}_2. \tag{A.15}$$

Note that this is not a proper Lorentz rotation, because $\boldsymbol{\gamma}_2$ is not an element of the group $\mathsf{SL}(2)$.

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